

# Pfaffian structures and certain solutions to BKP hierarchies I. Sums over partitions

A. Yu. Orlov\*

T. Shiota†

K. Takasaki‡

January 24, 2012

## Abstract

We introduce a useful and rather simple class of BKP tau functions which we shall call “easy tau functions”. We consider two versions of BKP hierarchy, one we will call “small BKP hierarchy” (sBKP) related to  $O(\infty)$  introduced in [4] and “large BKP hierarchy” (lBKP) related to  $O(2\infty + 1)$  introduced in [13] (which is closely related to the large  $O(2\infty)$  DKP hierarchy (lDKP) introduced in [6]). Actually “easy tau functions” of the sBKP hierarchy were already considered in [26], here we are more interested in the lBKP case and also the mixed small-large BKP tau functions [13]. Tau functions under consideration are equal to certain sums over partitions and to certain multi-integrals over cone domains. In this way they may be applicable in models of random partitions and models of random matrices. Here is the first part of the paper where sums of Schur and projective Schur functions over partitions are considered.

**Key words:** integrable systems, Pfaffians, symmetric functions, Schur and projective Schur functions, random partitions, random matrices, orthogonal ensemble, symplectic ensembles

## 1 Introduction

In the seminal papers of Kyoto school KP hierarchies related to different symmetry groups were introduced. In such a way DKP and BKP hierarchies appeared as KP hierarchies related to  $D_\infty$  and  $B_\infty$  type root systems, while the original KP hierarchy of Druryuma-Zakharov-Shabat was assigned to the root system of type  $A_\infty$ . However different realizations of these hierarchies are possible. The BKP and DKP hierarchies presented in [4] are subhierarchies of the standard KP one: it is related to  $O(\infty)$  subgroup of  $GL(\infty)$  symmetry group of KP. Authors of [13] refer these BKP and DKP hierarchies as respectively neutral BKP and DKP hierarchies. We shall call them respectively small BKP (sBKP) and small DKP (sDKP) hierarchies. There is also different DKP hierarchy presented in the paper [5]<sup>1</sup> and related to  $O(2\infty) \supset GL(\infty)$  which contains KP as the subhierarchy. We shall this hierarchy as large DKP one (lDKP). At last in [13] the BKP hierarchy related to  $O(2\infty + 1) \supset GL(\infty)$  which also contains KP as a subhierarchy was introduced. We shall refer it as large BKP hierarchy<sup>2</sup> These “large” hierarchies are rather interesting and much less studied than the “small” ones. In [5] and [13] the fermionic representation for sDKP, sBKP, lDKP and lBKP tau function was written down and bilinear equation (‘Hirota equations’) were presented. The tau function of these hierarchies appeared in a number of various problems. In the paper [14] it was shown that under certain restrictions lBKP (and lDKP) tau functions coincide with the so-called Pfaff lattice tau function [3] which in particular describes the orthogonal ensemble of random matrices. In [14] nice fermionic representations for orthogonal and for symplectic ensembles were found and in this way it was shown that the partition function of these ensembles are

\*Institute of Oceanology, Nahimovskii Prospekt 36, Moscow 117997, Russia, email: orlovs55@mail.ru

†Kyoto University, Kyoto, Japan, Email address: shiota@math.kyoto-u.ac.jp

‡Graduate School of Human and Environmental Studies, Kyoto University, Yoshida, Sakyo, Kyoto, 606-8501, Japan E-mail address: takasaki@math.h.kyoto-u.ac.jp

<sup>1</sup>We mean the hierarchy referred in [5], [6] as the  $D'_\infty$  one.

<sup>2</sup>In [13] the large DKP and the large BKP hierarchies were called fermionic (also charged) DKP and BKP ones, and the small DKP and BKP hierarchies were called neutral DKP and BKP ones; we found these names a little bit misleading

examples of IBKP tau function. In the recent paper [24] the coupled "large" 2-DKP hierarchy was introduced and the quasi-classical limit of the IDKP hierarchy and of the 2-IDKP hierarchy was studied.

General solutions of IDKP and IBKP hierarchy may be found as solutions to Hirota-type equations [6], [13]<sup>3</sup>; Hirota equations for 2-IBKP hierarchy were written down in [24].

In the present paper we shall study certain 'simple' classes of solutions of the IBKP and 2-IBKP hierarchies ("easy tau functions") singled out by equations (199),(200),(201) as it is explained in Section 5. Actually these tau functions on the one hand generalize two examples presented by J. van de Leur in [14] on the other hand generalize tau functions considered in [29] and called tau functions of hypergeometric type. We believe that such tau functions will have various applications. In addition we find its natural to consider certain solutions of the IBKP hierarchy coupled to sBKP one. Let us mark that special solutions of sBKP were studied in [10], [11], [31]. sBKP was used in studies of various random processes, see [61], [67], [68], [69].

A IBKP tau function depends on the same set of higher times  $\mathbf{t} = (t_1, t_2, \dots)$  as a KP tau function and on two discrete parameters (discrete times)  $l$  and  $l'$  (instead of one parameter in the KP case), and may be written in form of Schur function expansion

$$\tau_{ll'}(\mathbf{t}) = \sum_{\lambda \in P} s_{\lambda}(\mathbf{t}) \Pi_{\lambda}(l, l') \quad (1)$$

where  $\Pi_{\lambda}(l, l')$  are certain Pfaffians. In (1) sum runs over the set of all partitions denoted by  $P$ . In case of 2-IBKP hierarchy a typical series has the following form

$$\tau_{ll'}(\mathbf{t}, \bar{\mathbf{t}}) = \sum_{\lambda, \mu \in P} s_{\lambda}(\mathbf{t}) \Pi_{\lambda\mu}(l, l') s_{\mu}(\bar{\mathbf{t}}) \quad (2)$$

which is an analog of the Takasaki series for 2D TL hierarchy, which is

$$\tau_l(\mathbf{t}, \bar{\mathbf{t}}) = \sum_{\lambda, \mu \in P} s_{\lambda}(\mathbf{t}) \pi_{\lambda\mu}(l) s_{\mu}(\bar{\mathbf{t}}) \quad (3)$$

where  $\pi_{\lambda\mu}(l)$  are certain determinants.

In the present paper we shall derive (1) from the fermionic representation of the IBKP tau function given in [13] and consider a set of examples and applications. In particular we will introduce a certain class of IBKP tau functions which may be considered as a generalization of the hypergeometric function (compare to [30] and [31]) which depends on the IBKP higher times  $\mathbf{t} = (t_1, t_2, \dots)$  and a set of parameters denoted by  $U = \{U_m, m \in \mathbb{Z}\}$ ,

$$\tau(\mathbf{t}) = \sum_{\lambda \in P} e^{-U_{\lambda}} s_{\lambda}(\mathbf{t})$$

and more general tau functions, see Section 5. Here  $s_{\lambda}$  are the Schur functions,  $\mathbf{t}$  are IBKP higher times. The sum ranges over the set of all partitions denoted by  $P$ .

An example of such tau functions is

$$\tau(\mathbf{t}) = \sum_{\lambda \subset (m^N)} s_{\lambda}(\mathbf{t})$$

where the sum ranges over all Young diagrams  $\lambda$  which can be arranged into a  $m$  by  $N$  rectangular where  $m, N$  are given numbers. At first sight one can think that it is just a particular case of the well-known series for the KP tau functions [7], [6]

$$\tau(\mathbf{t}) = \sum_{\lambda} s_{\lambda}(\mathbf{t}) \pi_{\lambda}$$

This guess is not right: for simplicity take  $m = N = \infty$ ; then in KP case the numbers  $\pi_{\lambda}$  should solve Plucker relations while  $\pi_{\lambda}$  all equal to 1 do not solve.

---

<sup>3</sup> There is an interesting remark by J.Harnad that the Hirota equation for IDKP and IBKP may be treated as analogues of the Plucker relations for isotropic Grassmannians called Cartan relations [15], see [16] for more details on the topic of Cartan relations.

Another interesting example of the IBKP tau function is as follows. Consider a subset of all partitions  $\mathbf{P}$  denoted  $\mathbf{FP}$  ("fat partitions") which consists of all partitions of even length of form  $(\lambda_1, \lambda_1, \lambda_2, \lambda_2, \lambda_3, \lambda_3, \dots) = \lambda \cup \lambda$ . Then

$$\tau(\mathbf{t}) = \sum_{\lambda \in \mathbf{FP}} e^{-U_\lambda} s_\lambda(\mathbf{t})$$

which is also an example of the IBKP tau function and which we will relate to a discrete version of  $\beta = 4$  ensembles. This tau function will be used in Section 7 in a problem related to random motion.

We also present different examples of tau functions which are written as multiple integrals over a cone domain. Such integrals appear in the theory of random matrices [17], [89]. Let us point out the pioneer paper [2], [3] which relates the orthogonal ensemble to Pfaff lattice and also the very helpful paper by J/van de Leur who have shown that both ensembles of real  $\beta = 1$  and  $\beta = 4$  are examples of IBKP theory. In [1] we modify some results of [14] to the case of sBKP and considered also the cases of three different  $\beta = 1$ ,  $\beta = 2$ , and  $\beta = 4$  ensembles

We shall explain what is the meaning of "independent variables"  $U = \{U_i\}$  and what are equations with respect to these variables. It is suitable to parameterize  $U$  by new variables  $\mathbf{t}^*$ , see (105) and (106), then for certain specifications of  $\mathbf{t}$  (see (99)- (102)) we find that

$$\tau(\mathbf{t}, U(\mathbf{t}^*)) = \sum_{\lambda \in \mathbf{P}} e^{-U_\lambda(\mathbf{t}^*)} s_\lambda(\mathbf{t})$$

are again IBKP tau functions now with respect to parameters  $\mathbf{t}^*$  which play the role of higher times. Let us mark that this tau function turned out to be a partition function for a model of random turn motion of vicious walkers introduced by M.Fisher [70]. This problem is considered in Section 7.

These times may be also considered as group times for convolution flows [22] and related to the action of vertex operators. Hamiltonians of these flows act in a diagonal way on the basis of Schur functions<sup>4</sup>.

These convolution flows on arbitrary IBKP tau function may be also interpreted in terms of 'dual' multisoliton IBKP tau functions whose higher times  $\mathbf{t}^*$  are related to parameters  $U = U(\mathbf{t}^*)$  mentioned above (see Section 6). This link between two IBKP tau functions is quite similar to the case studied in KP where such link between two tau functions was used for technical purposes in papers [47] and [56] and was clarified in [27] and in [23].

We found it is pertinent to present certain small BKP tau functions such as

$$\tau(\mathbf{t}', U) = \sum_{\lambda \in \mathbf{DP}} e^{-U_\lambda} Q_\lambda(\tfrac{1}{2}\mathbf{t}')$$

and also IBKP tau functions coupled to sBKP tau functions (Section ref), the simplest example

$$\tau(\mathbf{t}, \mathbf{t}') = \sum_{\substack{\lambda \in \mathbf{P} \\ \ell(\lambda) \leq N}} s_\lambda(\mathbf{t}) Q_{\lambda^-}(\tfrac{1}{2}\mathbf{t}')$$

where  $Q_{\lambda^-}$  are projective Schur functions,  $\lambda^-$  denotes a strict partition whose parts are shifted parts of a partition  $\lambda$ . This expression is a IBKP tau function with respect to the set  $\mathbf{t} = (t_1, t_2, \dots)$  of higher times. At the same time it is sBKP tau function with respect to the times  $\mathbf{t}'$ .

## 2 Sums of Schur functions

**Subsets of partitions.** In the following, we consider sums over partitions and strict partitions, which will be denoted by Greek letters  $\alpha, \beta$ . Recall [18] that a strict partition  $\alpha$  is a set of integers (parts)  $(\alpha_1, \dots, \alpha_k)$  with  $\alpha_1 > \dots > \alpha_k \geq 0$ . The length of a partition  $\alpha$ , denoted  $\ell(\alpha)$ , is the number of non-vanishing parts, thus it is either  $k$  or  $k - 1$ .

Let  $\mathbf{P}$  be the set of all partitions. We shall need two special subsets of  $\mathbf{P}$ .

The first one consists of all partitions  $\lambda = (\lambda_1, \dots, \lambda_{2n})$ ,  $0 \leq n \in \mathbb{Z}$ ,  $\lambda_{2n} \geq 0$ , which satisfy

$$\lambda_i + \lambda_{2n+1-i} \text{ is independent of } i, \quad i = 1, \dots, 2n,$$

---

<sup>4</sup>First similar Hamiltonians were considered in the study of generalized Kontsevich model in [40].

or equivalently

$$h_i + h_{2n+1-i} = 2c \text{ is independent of } i \text{ (hence } h_1 + h_{2n} \geq 2n - 1), \quad i = 1, \dots, 2n, \quad (4)$$

where  $h_i = \lambda_i - i + 2n$ , and  $2c$  is a natural number conditioned by  $2c \geq 2n$ . This subset consists of all partitions  $\lambda$  of length  $l(\lambda) \leq 2n$  whose Young diagram satisfies the property that its complement in the rectangular Young diagram  $\bar{Y}$  corresponding to  $(\lambda_1 + \lambda_{2n})^{2n}$  coincides with itself rotated 180 degrees around the center of  $\bar{Y}$ . This set of partitions will be denoted by SCP( $c$ ) or simply SCP, for “self-complementary partitions”. If we introduce

$$y_i := h_i - c, \quad c = \frac{h_1 + h_{2n-1}}{2}, \quad (5)$$

then relation (4) may be rewritten as

$$y_i + y_{2n+1-i} = 0. \quad (6)$$

The second subset we need consists of the partitions  $\lambda$  which satisfy, equivalently,

$$\lambda_{2i} = \lambda_{2i-1}, \quad i = 1, 2, \dots, \quad (7)$$

or  $\lambda = \mu \cup \mu := (\mu_1, \mu_1, \mu_2, \mu_2, \dots, \mu_k, \mu_k)$  ( $\exists \mu = (\mu_1, \mu_2, \dots, \mu_k) \in P$ ), or that the conjugate partitions of  $\lambda$  are even, i.e., the ones whose parts are even numbers. This set of partitions will be denoted by FP, for “fat partitions”.

Following [18] we will denote by DP the set of all strict partitions (partitions with distinct parts), namely, partitions  $(\alpha_1, \alpha_2, \dots, \alpha_k)$ ,  $1 \leq k \in \mathbb{Z}$  with the strict inequalities  $\alpha_1 > \alpha_2 > \dots > \alpha_k > 0$ .

Strict partitions  $\alpha$  with the property

$$\alpha_{2i} = \alpha_{2i-1} + 1 \quad \text{for } 2i - 1 \leq l(\alpha), \quad (8)$$

where we set  $\alpha_{2i} = 0$  if  $l(\alpha) = 2i - 1$ , will be called fat strict partitions. The set of all fat strict partitions will be denoted by FDP.<sup>5</sup>

The set of all self-complementary strict partitions will be denoted by SCDP.

Let  $R_{NM}$  denote the set of all partitions whose Young diagram may be placed into the rectangle  $N \times M$ , namely,  $R_{NM}$  is the set of all partitions  $\lambda$  restricted by the conditions  $\lambda_1 \leq M$  and  $\ell(\lambda) \leq N$ .

**Sums over partitions.** Consider the following sums (for  $\mathbf{t} := (t_1, t_3, \dots)$ ,  $\mathbf{t}^* := (t_1^*, t_3^*, \dots)$ ,  $\bar{\mathbf{t}} := (\bar{t}_1, \bar{t}_3, \dots)$ ,  $\mathbf{t} := (t_1, t_3, \dots)$ ,  $\mathbf{t}^* := (t_1^*, t_3^*, \dots)$ ,  $\bar{\mathbf{t}} := (\bar{t}_1, \bar{t}_3, \dots)$ ),  $N$ ).

$$S^{(1)}(\mathbf{t}, N; U, \bar{A}) := \sum_{\substack{\lambda \in P \\ \ell(\lambda) \leq N}} \bar{A}_{h(\lambda)} e^{-U_{\{h\}}} s_{\lambda}(\mathbf{t}) \quad (9)$$

where  $h(\lambda) = \lambda_i - i + N$ . The factors  $\bar{A}_h$  on the right-hand side of (9) are determined in terms a pair  $(A, a) =: \bar{A}$  where  $A$  is an infinite skew symmetric matrix and  $a$  an infinite vector. For a strict partition  $h = (h_1, \dots, h_N)$ , the numbers  $\bar{A}_h$  are defined as the Pfaffian of an antisymmetric  $2n \times 2n$  matrix  $\tilde{A}$  as follows:

$$\bar{A}_h := \text{Pf}[\tilde{A}] \quad (10)$$

where for  $N = 2n$  even

$$\tilde{A}_{ij} = -\tilde{A}_{ji} := A_{h_i, h_j}, \quad 1 \leq i < j \leq 2n \quad (11)$$

and for  $N = 2n - 1$  odd

$$\tilde{A}_{ij} = -\tilde{A}_{ji} := \begin{cases} A_{h_i, h_j} & \text{if } 1 \leq i < j \leq 2n - 1 \\ a_{h_i} & \text{if } 1 \leq i < j = 2n. \end{cases} \quad (12)$$

In addition we set  $\bar{A}_0 = 1$ .

Then

$$U_{\{h\}} := \sum_{i=1}^N U_{h_i} \quad (13)$$

---

<sup>5</sup>This subset was used in [26] where it was denoted by DP’.

where  $U_n$ ,  $n = 0, 1, 2, \dots$  is a set of given complex numbers. This set is denoted by  $U$ .

As we see the factor  $e^{-U_{\{h\}}}$  can be included into the factor  $\bar{A}_h$  by redefinition of the data  $\bar{A}$  as follows:

$$A_{nm} \rightarrow A_{nm} e^{-U_n - U_m}, \quad a_n \rightarrow a_n e^{-U_n}$$

However we prefer to keep  $U$  as a set of parameters.

**Example 0** We choose the following matrix  $A$  is given by

$$A_{ik} = (A_0)_{ik} := \begin{cases} \text{sgn}(i - k) & \text{if } 1 \leq i, k \leq L \\ 0 & \text{otherwise} \end{cases}, \quad a_k = \begin{cases} 1 & \text{if } k \leq L \\ 0 & \text{otherwise} \end{cases}. \quad (14)$$

*Remark 1.* The matrix  $A_1$  is infinite. However if in series (9) we put  $U_n = +\infty$  for  $n > L$ , it will be the same as if we deals with the finite  $L$  by  $L$  matrix  $A$ , given by (14).

**Example 1**

$$A_{ik} = (A_1)_{ik} := 1, \quad i < k, \quad a_k = 1 \quad (15)$$

Then

$$(\bar{A}_1)_{\{h\}} = 1 \quad (16)$$

**Example 2**

The matrix  $A$  is a finite  $2n$  by  $2n$  matrix, and  $a = 0$ , thus the sum (9) ranges only partitions with even number of non-vashing parts. We put

$$A_{ik} = (A_2)_{ik} := -\delta_{i, 2c-i}, \quad i < k \quad (17)$$

Then

$$(\bar{A}_2)_{\{h\}} = \begin{cases} 1 & \text{iff } \lambda \in \text{SCP}(c) \\ 0 & \text{otherwise} \end{cases} \quad (18)$$

where  $h = (h_1, \dots, h_N)$  is related to  $\lambda = (\lambda_1, \dots, \lambda_N)$  as  $h_i = \lambda_i - i + N, i = 1, \dots, N = 2n$ .

**Example 3** Given set of additional variables  $\mathbf{t}' = (t'_1, t'_3, t'_5, \dots)$  where we take

$$A_{nm} = (A_3)_{nm} := \frac{1}{2} e^{-U_m - U_n} Q_{(n,m)}(\frac{1}{2}\mathbf{t}'), \quad a_n = (a_3)_n := e^{-U_n} Q_{(n)}(\frac{1}{2}\mathbf{t}') \quad (19)$$

Here, the *projective Schur functions*  $Q_\alpha$  are weighted polynomials in the variables  $t'_m$ ,  $\deg t'_m = m$ , labeled by strict partitions (See [18] for their detailed definition.)

*Remark 2.* Let us introduce notation  $\mathbf{t}'_\infty = (1, 0, 0, \dots)$ . It is known that  $Q_h(\frac{1}{2}\mathbf{t}'_\infty) = \Delta^*(h) \prod_{i=1}^N \frac{1}{h_i!}$  where

$$\Delta^*(h) := \prod_{i < j} \frac{h_i - h_j}{h_i + h_j} \quad (20)$$

Thus for this choice of  $\mathbf{t}'$  we obtain

$$(\bar{A}_3)_{\{h\}} = \Delta^*(h) \prod_{i=1}^N \frac{1}{h_i!} \quad (21)$$

One may compare it with Example 5 where  $f(n) = n$ .

**Example 4**

$$A_{nm} = (A_4)_{nm} := \delta_{n+1, m} - \delta_{m+1, n}. \quad (22)$$

Then

$$(\bar{A}_4)_{\{h\}} = \begin{cases} 1 & \text{iff } \lambda = (\lambda_1, \dots, \lambda_{2n}) \in \text{FP} \\ 0 & \text{otherwise} \end{cases} \quad (23)$$

where  $h = (h_1, \dots, h_N)$  is related to  $\lambda = (\lambda_1, \dots, \lambda_N)$  as  $h_i = \lambda_i - i + N, i = 1, \dots, N = 2n$ .

*Remark 3.* For some applications we may need further examples. In Examples 5-7  $\bar{A}$  depends on a given function on the lattice denoted by  $f$ . In particular one can choose  $f(n) = n$ . Below are examples of matrices  $A$  whose Pfaffians are well-known (see [94] and references there).

**Example 5**

$$A_{nm} = (A_5)_{nm} := \frac{f(n) - f(m)}{f(n) + f(m)} \quad (24)$$

Then for  $h_i = \lambda_i - i + N, i = 1, \dots, N$ , we have

$$(\bar{A}_5)_{\{h\}} = \Delta_N^{(5)}(f(h)) \quad (25)$$

where

$$\Delta_N^{(5)}(f(h)) := \prod_{i < j \leq N} \frac{f(h_i) - f(h_j)}{f(h_i) + f(h_j)} \quad (26)$$

**Example 6**

$$A_{nm} = (A_6)_{nm} := \frac{f(n) - f(m)}{1 - f(n)f(m)} \quad (27)$$

Then for  $h_i = \lambda_i - i + N, i = 1, \dots, N$ , we have

$$(\bar{A}_6)_{\{h\}} = \Delta_N^{(6)}(f(h)) \quad (28)$$

where

$$\Delta_N^{(6)}(f(h)) := \prod_{i < j \leq N} \frac{f(h_i) - f(h_j)}{1 - f(h_i)f(h_j)} \quad (29)$$

**Example 7**

$$A_{nm} = (A_7)_{nm} := \frac{f(n) - f(m)}{(f(n) + f(m))^2}. \quad (30)$$

Then for  $h_i = \lambda_i - i + N, i = 1, \dots, N$ , we have

$$(\bar{A}_7)_{\{h\}} = \Delta_N^{(7)}(f(h)) \quad (31)$$

where

$$\Delta_N^{(7)}(f(h)) := \left( \prod_{i < j \leq N} \frac{f(h_i) - f(h_j)}{(f(h_i) + f(h_j))^2} \right) \text{Hf} \left( \frac{1}{f(h_i) + f(h_j)} \right) \quad (32)$$

Having these examples we introduce the notation

$$S_i^{(1)}(\mathbf{t}, N; U) := S^{(1)}(\mathbf{t}, N; U, \bar{A}_i) = \sum_{\substack{\lambda \in \mathbb{P} \\ \ell(\lambda) \leq N}} (\bar{A}_i)_{h(\lambda)} e^{-U_\lambda} s_\lambda(\mathbf{t}), \quad i = 1, \dots, 6 \quad (33)$$

In particular we obtain

$$S_0^{(1)}(\mathbf{t}, N; M, U) := \sum_{\lambda \in R_{N, M}} e^{-U_\lambda} s_\lambda(\mathbf{t}) \quad (34)$$

$$S_1^{(1)}(\mathbf{t}, N; U) := \sum_{\substack{\lambda \in \mathbb{P} \\ \ell(\lambda) \leq N}} e^{-U_\lambda} s_\lambda(\mathbf{t}) \quad (35)$$

$$S_2^{(1)}(\mathbf{t}, N; U, c) := \sum_{\substack{\lambda \in \text{SCP}(c) \\ \ell(\lambda) \leq N}} e^{-U_\lambda} s_\lambda(\mathbf{t}) \quad (36)$$

$$S_3^{(1)}(\mathbf{t}, N, \mathbf{t}'; U) := \sum_{\substack{\lambda \in \mathbb{P} \\ \ell(\lambda) \leq N}} e^{-U_\lambda} Q_{\alpha(\lambda)}(\tfrac{1}{2}\mathbf{t}') s_\lambda(\mathbf{t}) \quad (37)$$

$$S_4^{(1)}(\mathbf{t}, N = 2n, U) := \sum_{\substack{\lambda \in \text{FP} \\ \ell(\lambda) \leq N}} e^{-U_\lambda} s_\lambda(\mathbf{t}) \quad (38)$$

$$S_i^{(1)}(\mathbf{t}, N; U, f) := \sum_{\substack{\lambda \in \mathbb{P} \\ \ell(\lambda) \leq N}} \Delta_N^{(i)}(f(h)) e^{-U_\lambda} s_\lambda(\mathbf{t}), \quad i = 5, 6, 7 \quad (39)$$

The coefficients  $U_{\{\alpha\}}$  are defined as

$$U_{\{\alpha\}} := \sum_{i=1}^k U_{\alpha_i}, \quad (40)$$

The notation  $U_\lambda$  serves for

$$U_\lambda := U_{\{h\}}, \quad h_i = \lambda_i - i + \ell(\lambda) \quad (41)$$

**Proposition 1.** Sums (9),(34)-(39) are tau functions of the “large” BKP hierarchy introduced in [13] with respect to the time variables  $\mathbf{t}$ . Sums (37) are tau functions of the BKP hierarchy introduced in [6] with respect to the time variables  $\mathbf{t}'$ .

**Sums over pairs of strict partitions.** In the Frobenius notations [18] we write  $\lambda = (\alpha|\beta) = (\alpha_1, \dots, \alpha_k | \beta_1, \dots, \beta_k)$ , where  $\alpha = (\alpha_1, \dots, \alpha_k)$ ,  $\alpha_1 > \dots > \alpha_k \geq 0$  and  $\beta = (\beta_1, \dots, \beta_k)$  may be viewed as strict partitions. It is clear that  $\ell(\alpha) = \ell(\beta), \ell(\beta) \pm 1$ , and we imply this restriction in sums over pairs of strict partitions below.

Now we consider

$$S^{(2)}(\mathbf{t}; U, \bar{A}, \bar{B}) := 1 + \sum_{\alpha, \beta \in \text{DP}} e^{U_{\{-\beta-1\}} - U_{\{\alpha\}}} \bar{A}_\alpha s_{(\alpha|\beta)}(\mathbf{t}) \bar{B}_\beta \quad (42)$$

where given infinite skew matrices  $A$  and  $B$  and given vectors  $a$  and  $b$ , the factors  $\bar{A}_\alpha$  and  $\bar{B}_\alpha$  are defined in the same way as before.

$$U_{\{\alpha\}} = \sum_{i=1}^k U_{\alpha_i}, \quad U_{\{-\beta-1\}} = \sum_{i=1}^k U_{-\beta_i-1} \quad (43)$$

We introduce the following notation

$$S_{ij}^{(2)}(\mathbf{t}; U) := S^{(2)}(\mathbf{t}; U, \bar{A}_i, \bar{A}_j) \quad (44)$$

where  $i, j = 1, \dots, 7$  and matrices  $A_i$  are taken from the Examples 1-7 above. In particular we obtain series

$$S_{11}^{(2)}(\mathbf{t}; U) := \sum_{\lambda \in \text{P}} e^{-U_\lambda} s_\lambda(\mathbf{t}) \quad (45)$$

$$S_{22}^{(2)}(\mathbf{t}; U) := 1 + \sum_{\alpha, \beta \in \text{SCDP}} e^{U_{\{\beta\}} - U_{\{\alpha\}}} s_{(\alpha|\beta)}(\mathbf{t}) \quad (46)$$

$$S_{24}^{(2)}(\mathbf{t}; U) := 1 + \sum_{\alpha \in \text{SCDP}, \beta \in \text{FDP}} e^{U_{\{\beta\}} - U_{\{\alpha\}}} s_{(\alpha|\beta)}(\mathbf{t}) \quad (47)$$

$$S_{31}^{(2)}(\mathbf{t}, \mathbf{t}'; U) := 1 + \sum_{\alpha, \beta \in \text{DP}} e^{U_{\{\beta\}} - U_{\{\alpha\}}} Q_{\bar{\alpha}(\alpha)}(\frac{1}{2}\mathbf{t}') s_{(\alpha|\beta)}(\mathbf{t}) \quad (48)$$

$$S_{41}^{(2)}(\mathbf{t}; U) := 1 + \sum_{\alpha \in \text{FDP}, \beta \in \text{DP}} e^{U_{\{\beta\}} - U_{\{\alpha\}}} s_{(\alpha|\beta)}(\mathbf{t}) \quad (49)$$

$$S_{44}^{(2)}(\mathbf{t}; U) := 1 + \sum_{\alpha, \beta \in \text{FDP}} e^{U_{\{\beta\}} - U_{\{\alpha\}}} s_{(\alpha|\beta)}(\mathbf{t}) \quad (50)$$

$$S_{33}^{(2)}(\mathbf{t}, \mathbf{t}', \mathbf{t}''; U) := 1 + \sum_{\alpha, \beta \in \text{DP}} e^{U_{\{\beta\}} - U_{\{\alpha\}}} Q_{\bar{\alpha}(\alpha)}(\frac{1}{2}\mathbf{t}') s_{(\alpha|\beta)}(\mathbf{t}) Q_{\bar{\beta}(\beta)}(\frac{1}{2}\mathbf{t}'') \quad (51)$$

$$S_{34}^{(2)}(\mathbf{t}, \mathbf{t}'; U) := 1 + \sum_{\alpha \in \text{DP}, \beta \in \text{FDP}} e^{U_{\{\beta\}} - U_{\{\alpha\}}} Q_{\bar{\alpha}(\alpha)}(\frac{1}{2}\mathbf{t}') s_{(\alpha|\beta)}(\mathbf{t}) \quad (52)$$

$$S_{ij}^{(2)}(\mathbf{t}; U, f) := 1 + \sum_{\alpha \in \text{DP}, \beta \in \text{DP}} e^{U_{\{\beta\}} - U_{\{\alpha\}}} \Delta^{(i)}(f(\alpha)) s_{(\alpha|\beta)}(\mathbf{t}) \Delta^{(j)}(f(\beta)), \quad i, j = 5, 6, 7 \quad (53)$$

Each  $Q_\alpha(\frac{1}{2}\mathbf{t}')$  is known to be a BKP [4], [6] tau function. (This was a nice observation of [10], [11]). The fact that only odd subscripts appear in the BKP higher times  $t_{2m-1}$  is related to the reduction from the KP hierarchy.

**Proposition 2.** Sums (42),(44) are tau functions of the “large” BKP hierarchy introduced in [13] with respect to the time variables  $\mathbf{t}$ . Sums (48) are tau functions of the BKP hierarchy introduced in [6] with respect to the time variables  $\mathbf{t}'$ . Sums (51) are tau functions of the two-component BKP hierarchy introduced in [6] with respect to the time variables  $\mathbf{t}'$  and  $\mathbf{t}''$ .

*Remark 4.* Let us remind that for the small BKP hierarchy obtained from KP we have the following [26]

$$S = \sum_{\alpha \in \text{DP}} \bar{A}_\alpha Q_\alpha(\mathbf{t}') \quad (54)$$

By specification of the data  $\bar{A}$  we obtain

$$\sum_{\alpha \in \text{DP}} e^{-U_{\{\alpha\}}} Q_\alpha(\tfrac{1}{2}\mathbf{t}'), \quad \sum_{\alpha \in \text{DP}} e^{-U_{\{\alpha\}}} Q_\alpha(\tfrac{1}{2}\mathbf{t}') Q_\alpha(\tfrac{1}{2}\mathbf{t}''), \quad \sum_{\alpha \in \text{DP}'} e^{-U_{\{\alpha\}}} Q_\alpha(\tfrac{1}{2}\mathbf{t}') \quad (55)$$

The sums (55) are particular examples (see [26]) of BKP tau functions, as introduced in [4], defining solutions to what was called the small BKP hierarchy in [13].

The coupled small BKP yields series

$$S_5(\mathbf{t}', \mathbf{t}'', D) := \sum_{\substack{\alpha, \beta \in \text{DP} \\ \ell(\alpha) = \ell(\beta)}} Q_\alpha(\tfrac{1}{2}\mathbf{t}') D_{\alpha, \beta} Q_\beta(\tfrac{1}{2}\mathbf{t}'') \quad (56)$$

The coefficients  $D_{\alpha, \beta}$  in (56) are defined as determinants:

$$D_{\alpha, \beta} = \det (D_{\alpha_i, \beta_j}) \quad (57)$$

where  $D$  is a given infinite matrix. Taking  $D_{nm} = e^{U_m - U_n} s_{(n|m)}(\mathbf{t})$  we reproduce (51).

## 2.1 Action of $\Psi DO$ algebra on sums

Here we shall describe certain group transformation properties of sums  $S^{(1)}$  and  $S^{(2)}$ . We shall also present some Virasoro invariant sums  $S^{(1)}$ .

Consider the following operator acting on the space of functions of infinitely many variables  $\mathbf{t} = (t_1, t_2, \dots)$

$$\hat{W}_k^{(g)} := \text{res}_x x^k (g(D_x) \cdot Z(y, x))|_{y=x} \quad (58)$$

where  $Z(x, y)$  is the vertex operator [6]

$$Z(y, x) := \left( e^{\sum_{n=1}^{\infty} (y^n - x^n) t_n} e^{\sum_{n=1}^{\infty} (\frac{1}{n x^n} - \frac{1}{n y^n}) \frac{\partial}{\partial t_n}} - 1 \right) \sum_{n=0}^{+\infty} \frac{x^n}{y^{n+1}} \quad (59)$$

and where  $D_x := x \partial_x + \frac{1}{2}$  is the Euler operator and  $g$  is a given function of one variable. We assume that  $g(D_x) \cdot x^n = g(n + \frac{1}{2}) x^n$ .

The operators  $\hat{W}_k^{(g)}$  act as symmetry transformation generators on tau functions, and this action may be embedded into the algebra of infinite matrices with the central extension, see [6] and references therein. The matrix which corresponds to  $\hat{W}_k^{(g)}$  is as follows

$$W_k^{(g)} := \Lambda^k g(D), \quad (\Lambda)_{nm} = \delta_{n, m-1}, \quad (D)_{nm} = \left( n + \frac{1}{2} \right) \delta_{n, m} \quad (60)$$

*Remark 5.* Operators  $\hat{W}_k^{(g)}$  may be also viewed as the element  $x^k g(D_x)$  of the algebra of pseudodifferential operators ( $\Psi DO$ ) on the circle with a central extension, see Appendix D

In the Proposition below we shall use the notation  $(A)_-$  to denote the antisymmetric part of a matrix  $A$ .

**Proposition 3.** For any data  $\bar{A} = (A, a)$ ,  $\bar{B} = (B, b)$  where  $A, B$  are (infinite) antisymmetric matrices and  $a, b$  are (infinite) column vectors, consider the following one-parameter family of

$$\bar{A}(U, \mathbf{t}) := \left( \left( e^{t W_k^{(g)}} e^U A e^U e^{t W_k^{(g)}} \right)_-, e^{t W_k^{(g)}} e^U a \right), \quad \bar{B}(U, \mathbf{t}) := \left( \left( e^{t W_k^{(g)}} e^U B e^U e^{t W_k^{(g)}} \right)_-, e^{t W_k^{(g)}} e^U b \right) \quad (61)$$

where  $U = \text{diag}(U_n)$  and where we assume that matrices and vectors in the right hand sides of equalities do exist as formal series in a parameter  $\mathbf{t}$ . Then

$$e^{t \hat{W}_k^{(g)}} \cdot S^{(1)}(\mathbf{t}, N, U, \bar{A}) = S^{(1)}(\mathbf{t}, N, 0, \bar{A}(U, \mathbf{t})), \quad k > 0 \quad (62)$$

$$e^{t \hat{W}_k^{(g)}} \cdot S^{(2)}(\mathbf{t}, N, U, \bar{A}, \bar{B}) = S^{(2)}(\mathbf{t}, N, 0, \bar{A}(U, \mathbf{t}), \bar{B}(U, \mathbf{t})), \quad k > 0 \quad (63)$$

where  $\hat{W}_k^{(g)}$  and  $W_k^{(g)}$  are given respectively by (58) and (60), and the exponential  $e^{t \hat{W}_k^{(g)}}$  is considered as formal Taylor series in the parameter  $\mathbf{t}$ .



## 2.2 Pfaffian representations

For

$$\mathbf{t} = \mathbf{t}(\mathbf{x}^{(M)}) =: [x_1] + \cdots + [x_M] \quad (64)$$

we have for any  $N \geq M = 1$

$$S^{(1)}(\mathbf{t}(x_i); N, U, \bar{A}) = \sum_{n=0}^{\infty} a_n e^{-U_n} x_i^n \quad (65)$$

and for any  $N \geq M = 2$  we have

$$S^{(1)}(\mathbf{t}(x_i, x_j); N, U, \bar{A}) = \frac{1}{x_i - x_j} \sum_{m>n \geq 0}^{\infty} A_{nm} e^{-U_n - U_m} (x_i^m x_j^n - x_i^n x_j^m) \quad (66)$$

**Proposition 4.** For  $M = N$  we have

$$S^{(1)}(\mathbf{t}(\mathbf{x}^{(M)}); N, U, \bar{A}) = \frac{1}{\Delta_N(x)} \text{Pf}[\tilde{S}] \quad (67)$$

where for  $N = 2n$  even

$$\tilde{S}_{ij} = -\tilde{S}_{ji} := (x_i - x_j) S^{(1)}(\mathbf{t}(x_i, x_j), N, U, \bar{A}), \quad 1 \leq i < j \leq 2n \quad (68)$$

and for  $N = 2n - 1$  odd

$$\tilde{S}_{ij} = -\tilde{S}_{ji} := \begin{cases} (x_i - x_j) S^{(1)}(\mathbf{t}(x_i, x_j), N, U, \bar{A}) & \text{if } 1 \leq i < j \leq 2n - 1 \\ S^{(1)}(\mathbf{t}(x_i), N, U, \bar{A}) & \text{if } 1 \leq i < j = 2n \end{cases} \quad (69)$$

and where

$$\Delta_N(x) := \prod_{0 \leq i < j \leq N} (x_i - x_j) \quad (70)$$

We shall omit more spacious formulae for the case  $M \neq N$ .

*Remark 6.* Let us write down the entries of  $\tilde{S}$  to express  $S_i^{(1)}$ ,  $i = 0, \dots, 6$

$$S_1^{(1)}(\mathbf{t}(x_i, x_j), N, U) = (x_i - x_j)^{-1} \sum_{m>n \geq 0}^{\infty} e^{-U_n - U_m} (x_i^m x_j^n - x_j^m x_i^n) \quad (71)$$

$$S_1^{(1)}(\mathbf{t}(x_i), N, U) = \sum_{n=0}^{\infty} e^{-U_n} x_i^n \quad (72)$$

$$S_2^{(1)}(\mathbf{t}(x_i, x_j), 2n, U) = (x_i - x_j)^{-1} \sum_{n=0}^{\infty} e^{-U_n - U_{c-n}} (x_i^{c-n} x_j^n - x_j^{c-n} x_i^n) \quad (73)$$

$$S_3^{(1)}(\mathbf{t}(x_i, x_j), N, U) = (x_i - x_j)^{-1} \sum_{m>n \geq 0}^{\infty} e^{-U_n - U_m} Q_{(n,m)}(\mathbf{t}') (x_i^n x_j^m - x_j^n x_i^m) \quad (74)$$

$$S_3^{(1)}(\mathbf{t}(x_i), N, U) = \sum_{n \geq 0}^{\infty} e^{-U_n} Q_{(n)}(\mathbf{t}') x_i^n \quad (75)$$

$$S_4^{(1)}(\mathbf{t}(x_i, x_j), N^{\epsilon}, U) = (x_i - x_j)^{-1} \sum_{n \geq 0}^{\infty} e^{-U_n - U_{n+1}} (x_i^n x_j^{n+1} - x_j^n x_i^{n+1}) \quad (76)$$

$$S_4^{(1)}(\mathbf{t}(x_i), N, U) = \sum_{n \geq 0}^{\infty} e^{-U_n} x_i^n \quad (77)$$

In particular substituting (15), (22) we obtain

$$S_1^{(1)}(\mathbf{t}, N, U = 0) = \frac{1}{\Delta_N(x)} \text{Pf} \frac{x_j - x_i}{(1 - x_i)(1 - x_j)(1 - x_i x_j)} \quad (78)$$

$$S_4^{(1)}(\mathbf{t}, N, U = 0) = \frac{1}{\Delta_N(x)} \text{Pf} \frac{x_j - x_i}{1 - x_i x_j} \quad (79)$$

Then it follows that

$$\sum_{\lambda \in \mathbf{P}} s_\lambda(\mathbf{t}(\mathbf{x}^N)) = \prod_{i=1}^N (1 - x_i)^{-1} \prod_{i < j \leq N} (1 - x_i x_j)^{-1} \quad (80)$$

and

$$\sum_{\lambda \in \mathbf{P}} s_{\lambda \cup \lambda}(\mathbf{t}(\mathbf{x}^N)) = \prod_{i < j \leq N} (1 - x_i x_j)^{-1} \quad (81)$$

Formulae (80) and (81) are known, see Ex-s 4-5 in I-5 of [18].

It is convenient to re-write these formulae in a way independent of the choice of  $N$ :

**Proposition 5.**

$$\sum_{\lambda \in \mathbf{P}} s_\lambda(\mathbf{t}) = e^{\frac{1}{2} \sum_{m=1}^{\infty} m t_m^2 + \sum_{m=1}^{\infty} t_{2m-1}} \quad (82)$$

and

$$\sum_{\lambda \in \mathbf{P}} s_{\lambda \cup \lambda}(\mathbf{t}) = e^{\frac{1}{2} \sum_{m=1}^{\infty} m t_m^2 + \sum_{m=1}^{\infty} t_{2m}} \quad (83)$$

Relations (82) and (83) will be used later in Section to solve certain combinatorial problem. From

$$s_{\lambda^{tr}}(\mathbf{t}) = (-1)^{|\lambda|} s_\lambda(-\mathbf{t}) \quad (84)$$

we obtain

$$\sum_{\lambda \in \mathbf{P}} (-1)^{|\lambda|} s_\lambda(\mathbf{t}) = e^{\frac{1}{2} \sum_{m=1}^{\infty} m t_m^2 - \sum_{m=1}^{\infty} t_{2m-1}} \quad (85)$$

and

$$\sum_{\lambda \in \mathbf{P}_{\text{even}}} s_\lambda(\mathbf{t}) = e^{\frac{1}{2} \sum_{m=1}^{\infty} m t_m^2 - \sum_{m=1}^{\infty} t_{2m}} \quad (86)$$

By the simple re-scaling  $t_m \rightarrow z^m t_m$  in equations (82)-(86) and equating factors before same powers of  $z$  we obtain

**Proposition 6.**

$$\sum_{\substack{\lambda \in \mathbf{P} \\ |\lambda| = \tau}} s_\lambda(\mathbf{t}) = s_{(\tau)}(\tilde{\mathbf{t}}) \quad \left[ \tilde{t}_{2m-1} = t_{2m}, \tilde{t}_{2m} = \frac{m}{2} t_m^2 \right] \quad (87)$$

$$\sum_{\substack{\lambda \in \mathbf{P} \\ |\lambda| = \tau}} (-1)^{|\lambda|} s_\lambda(\mathbf{t}) = s_{(\tau)}(\tilde{\mathbf{t}}) \quad \left[ \tilde{t}_{2m-1} = -t_{2m}, \tilde{t}_{2m} = \frac{m}{2} t_m^2 \right] \quad (88)$$

$$\sum_{\substack{\lambda \in \mathbf{P} \\ |\lambda| = \tau}} s_{\lambda \cup \lambda}(\mathbf{t}) = s_{(\tau)}(\tilde{\mathbf{t}}) \quad \left[ \tilde{t}_m = \frac{m}{2} t_m^2 + t_{2m} \right] \quad (89)$$

$$\sum_{\substack{\lambda \in \mathbf{P}_{\text{even}} \\ |\lambda| = \tau}} s_\lambda(\mathbf{t}) = s_{(\tau)}(\tilde{\mathbf{t}}) \quad \left[ \tilde{t}_m = \frac{m}{2} t_m^2 - t_{2m} \right] \quad (90)$$

where auxiliary sets of times  $\tilde{\mathbf{t}} = (\tilde{t}_1, \tilde{t}_2, \dots)$  are specified in the brackets to the right of equalities.

For instance we get (87) from (82) using the equality

$$\sum_{\lambda \in \mathbf{P}} z^{|\lambda|} s_\lambda(\mathbf{t}) = e^{\sum_{m=1}^{\infty} \frac{z^{2m}}{2} m t_m^2 + \sum_{m=1}^{\infty} z^{2m-1} t_{2m-1}} = \sum_{\mathbf{T}=0}^{\infty} z^{\mathbf{T}} s_{(\mathbf{T})}(\tilde{\mathbf{t}})$$

where  $\tilde{\mathbf{t}} = \left( t_1, \frac{1 \cdot t_1^2}{2}, t_3, \frac{2 \cdot t_2^2}{2}, t_5, \frac{3 \cdot t_3^2}{2}, \dots \right)$ .

Formula (87) in case  $\mathbf{t} = (1, 0, 0, \dots)$  has an interpretation in terms of total numbers of standard tableaux of weight  $(1^T)$  and numbers of involutive permutations of  $S_T$ , see Ex 12 I.5 of [18], [76], [81] (see also (291)).

We get from Proposition 4

**Proposition 7.**

$$S_1^{(1)}(\mathbf{t}(\mathbf{x}^{(2n)}), 2n, U) = \frac{1}{\Delta_{2n}(x)} \text{Pf} \left[ \sum_{m > n \geq 0} e^{-U_n - U_m} (x_j^m x_i^n - x_i^m x_j^n) \right]_{i,j=1,\dots,2n} \quad (91)$$

Choosing  $U_m = 0$ ,  $m \leq L + 2n - 1$  and  $U_m = +\infty$ ,  $m > L + 2n - 1$  we obtain

$$\sum_{\substack{\lambda \in P \\ \lambda_1 \leq L}} s_\lambda(\mathbf{t}(\mathbf{x}^{(2n)})) = \frac{1}{\Delta_{2n}(x)} \text{Pf} \left[ \frac{x_j - x_i}{(1 - x_i)(1 - x_j)(1 - x_i x_j)} \left( 1 - (x_i x_j)^{L+N} + \frac{x_j^{L+N} - x_i^{L+N}}{x_j - x_i} \right) \right]_{i,j=1,\dots,2n} \quad (92)$$

**Proposition 8.**

$$S_4^{(1)}(\mathbf{t}(\mathbf{x}^{(2n)}), 2n, U) = \frac{1}{\Delta_{2n}(x)} \text{Pf} [(x_j - x_i) f(x_i x_j, U)]_{i,j=1,\dots,2n} \quad (93)$$

where

$$f(z, U) = \sum_{m=0}^{\infty} e^{-U_m - U_{m+1}} z^m = S_4^{(1)}(\mathbf{t}(x_i, x_j), U)$$

Choosing  $U_m = 0$ ,  $m \leq L + 2n - 1$  and  $U_m = +\infty$ ,  $m > L + 2n - 1$  we obtain

$$\sum_{\substack{\lambda \in P \\ \lambda_1 \leq L}} s_{\lambda \cup \lambda}(\mathbf{t}(\mathbf{x}^{(2n)})) = \frac{1}{\Delta_{2n}(x)} \text{Pf} \left[ (x_j - x_i) \frac{1 - (x_i x_j)^{L+2n-1}}{1 - x_i x_j} \right]_{i,j=1,\dots,2n} \quad (94)$$

Next, as a corollary of Proposition 4 we obtain

**Proposition 9.**

$$\sum_{\lambda \in \text{SCP}(c)} s_\lambda(\mathbf{t}(\mathbf{x}^{(2n)})) = \frac{1}{\Delta_{2n}(x)} \text{Pf} \left( \frac{\left( x_j^{[c+\frac{1}{2}]} - x_i^{[c+\frac{1}{2}]} \right)^2}{x_j - x_i} \right)_{i,j=1,\dots,2n} \quad (95)$$

$$\sum_{\lambda \in \text{SCP}(c)} (-1)^{\sum_{i=1}^n (\lambda_i - i + 2n)} s_\lambda(\mathbf{t}(\mathbf{x}^{(2n)})) = \frac{1}{\Delta_{2n}(x)} \text{Pf} \left( \left( x_j^{[c+\frac{1}{2}]} - x_i^{[c+\frac{1}{2}]} \right) \frac{x_j^{[c+\frac{1}{2}]} - (-x_i)^{[c+\frac{1}{2}]}}{x_j + x_i} \right)_{i,j=1,\dots,2n} \quad (96)$$

where  $[a]$  is equal to the integer part of  $a$ . Notice that in case  $c = n$  we have only one term related to  $\lambda = 0$  and thus the both sides of identity (95) are equal to 1 (compare to Lemma 5.7 in [94]).

## 2.3 Specializations and Examples

**Links with group characters** There is a known relation (see [8]) between the Schur functions and the odd orthogonal character  $so_\lambda$  of rectangular shape as follows

$$\sum_{\lambda_1 \leq p} s_\lambda(x_1, \dots, x_m) = (x_1 \dots x_m)^{\frac{1}{2}p} so\left(\left(\frac{p}{2}\right)^m\right)(x_1^{\pm 1}, \dots, x_m, x_m^{\pm 1}, 1) \quad (97)$$

The odd orthogonal characters  $so_\lambda(x_1^{\pm 1}, x_2^{\pm 1}, \dots, x_m^{\pm 1}, 1)$ , where  $x_1^{\pm 1}$  is a shorthand notation for  $x_1, x_1^{-1}, \dots$ , and where  $\lambda$  is an  $m$ -tuple  $(\lambda_1, \lambda_2, \dots, \lambda_m)$  of integers, or of half-integers, is defined by

$$so_\lambda(x_1^{\pm 1}, \dots, x_m^{\pm 1}, 1) := \frac{\det \left( x_j^{\lambda_i - i + m + \frac{1}{2}} - x_j^{-(\lambda_i - i + m + \frac{1}{2})} \right)}{\det \left( x_j^{-i + m + \frac{1}{2}} - x_j^{-(-i + m + \frac{1}{2})} \right)} \quad (98)$$

(see, say, (3.3) in [8]). Thus,  $S_1^{(1)}$  may be equated to a special character of the orthogonal group.

There is the similar link between  $S_4^{(1)}$  and a character of the symplectic group.

## Links between sums and matrix models I. Sums as perturbation series for matrix models.

This topic will be considered separately in [1].

**Links between sums and matrix models II. Discrete analogs of matrix models.** We know [32] few (basically three) ways to choose parameters  $\mathbf{t}$  in order to convert series in the Schur function  $s_\lambda(\mathbf{t})$  to discrete analogues of matrix integrals where integrals over eigenvalues are replaced by sums over integers. These are

$$(A1) \quad \mathbf{t} = \mathbf{t}_\infty := (1, 0, 0, \dots) \quad (99)$$

$$(A2) \quad \mathbf{t} = \mathbf{t}(a, x) := a[x] \quad (100)$$

$$(B1) \quad \mathbf{t} = \mathbf{t}(q) := (t_1(q), t_2(q), t_3(q), \dots), \quad t_m(q) := \frac{1}{m} \frac{1}{1 - q^m} \quad (101)$$

$$(B2) \quad \mathbf{t} = \mathbf{t}(a; q) := (t_1(a; q), t_2(a; q), t_3(a; q), \dots), \quad t_m(a; q) := \frac{1}{m} \frac{1 - q^{am}}{1 - q^m} \quad (102)$$

$$(C) \quad \mathbf{t} = \mathbf{t}(\mathbf{x}^N) := \sum_{i=1}^N [x_i] \quad (103)$$

The notations  $[x]$  and  $a[x]$  are standard in soliton theory and denotes Miwa variables

$$[x] = \left( x, \frac{x^2}{2}, \frac{x^3}{3}, \dots \right), \quad a[x] = \left( ax, a\frac{x^2}{2}, a\frac{x^3}{3}, \dots \right) \quad (104)$$

These specializations of  $\mathbf{t}$  variables will be refereed as respectively the cases (A),(B) and (C) below. See Appendix E.4 for cases (99)-(102).

We have the following observation

**Proposition 10.** Let us choose specializations of  $\mathbf{t}$  variables according to either (99) or (100) in sums (9) and put

$$U_n = U_n(\mathbf{t}^*, \bar{\mathbf{t}}^*) = U_n^{(0)} + \sum_{m=1}^{\infty} \left( \frac{an+b}{cn+d} \right)^m t_m^* + t_0^* \ln \left( \frac{an+b}{cn+d} \right) - \sum_{m=1}^{\infty} \left( \frac{an+b}{cn+d} \right)^{-m} \bar{t}_m^* \quad (105)$$

where  $U_n^{(0)}$  and  $a, b, c, d$ , are arbitrary parameters conditioned by  $ad - bc \neq 0$ . Then sums (9) are tau functions of the “large” BKP, introduced in [13] with respect to the time variables  $\mathbf{t}^* = (t_1^*, t_2^*, \dots)$ .

**Proposition 11.** Let us choose specializations of  $\mathbf{t}$  variables according to either (101) or (102) in sums (9) and put

$$U_n = U_n(\mathbf{t}^*, \bar{\mathbf{t}}^*, q) = U_n^{(0)} + \sum_{m=1}^{\infty} \left( \frac{aq^n + b}{cq^n + d} \right)^m t_m^* + t_0^* \ln \left( \frac{aq^n + b}{cq^n + d} \right) - \sum_{m=1}^{\infty} \left( \frac{aq^n + b}{cq^n + d} \right)^{-m} \bar{t}_m^* \quad (106)$$

where  $U_n^{(0)}$  and  $a, b, c, d$ , are arbitrary parameters. Then sums (9) are tau functions of the “large” BKP, introduced in [13] with respect to the time variables  $\mathbf{t}^* = (t_1^*, t_2^*, \dots)$ .

We have

$$s_\lambda(\mathbf{t}_\infty) = \Delta_N(h) \prod_{i=1}^N \frac{1}{h_i!}, \quad h = (h_1, \dots, h_N) \quad (107)$$

where  $h_i := \lambda_i - i + N$ .

In case  $\lambda = (\lambda_1, \dots, \lambda_{2n}) \in \text{SCP}$  (see (4)) then in the variables  $y_i$  introduced in (5) thanks to (6) we can write

$$s_\lambda(\mathbf{t}_\infty) = (\Delta_n(y^2))^2 \prod_{i=1}^n \frac{1}{(c - y_i)!(c + y_i)!}, \quad y = (y_1, \dots, y_n) \quad (108)$$

where  $y$  are related to  $\lambda$  as

$$y_i = \lambda_i - i + 2n - c, \quad i = 1, \dots, n \quad (109)$$

If  $\lambda$  is rewritten in the Frobenius notations,  $\lambda = (\alpha_i, \dots, \alpha_k | \beta_1, \dots, \beta_k)$ , then the last relation may be written as

$$s_{(\alpha|\beta)}(\mathbf{t}_\infty) = \frac{\Delta_k(\alpha)\Delta_k(\beta)}{\prod_{i,j=1}^k (\alpha_i + \beta_j + 1)} \prod_{i=1}^k \frac{1}{\alpha_i! \beta_i!} \quad (110)$$

We also have (see Remark 2)

$$Q_\alpha(\tfrac{1}{2}\mathbf{t}_\infty) = \Delta_k^{(3)}(\alpha) \prod_{i=1}^k \frac{1}{\alpha_i!}, \quad \alpha = (\alpha_1, \dots, \alpha_k) \quad (111)$$

where

$$\Delta_N(h) := \prod_{0 < i < j \leq N} (h_i - h_j), \quad \Delta_k^{(3)}(\alpha) := \prod_{0 < i < j \leq k} \frac{\alpha_i - \alpha_j}{\alpha_i + \alpha_j} \quad (112)$$

(A) First we choose the specialization (99). Then the parametrization (105) is available. Putting  $\mathbf{t} = \mathbf{t}_\infty$  we obtain from (9)

$$S^{(1)}(\mathbf{t}_\infty, N, U, \bar{A}) := \frac{1}{N!} \sum_{h_1, \dots, h_N=0}^M \bar{A}_{h(\lambda)} e^{-U_{\{h\}}(\mathbf{t}^*, \bar{\mathbf{t}}^*)} \Delta_N(h) \quad (113)$$

in particular

$$S_1^{(1)}(\mathbf{t}_\infty, N, U(\mathbf{t}^*, \bar{\mathbf{t}}^*)) := \frac{1}{N!} \sum_{h_1, \dots, h_N=0}^M |\Delta_N(h)| \prod_{i=1}^N \mu_1(h_i, \mathbf{t}^*, \bar{\mathbf{t}}^*) \quad (114)$$

$$S_2^{(1)}(\mathbf{t}_\infty, N = 2n, U(\mathbf{t}^*, \bar{\mathbf{t}}^*)) := \frac{1}{n!} \sum_{y_1, \dots, y_n=0}^M (\Delta_N(y^2))^2 \prod_{i=1}^n \mu_2(y_i, \mathbf{t}^*, \bar{\mathbf{t}}^*) \quad (115)$$

$$S_3^{(1)}(\mathbf{t}_\infty, N, \mathbf{t}_\infty, U(\mathbf{t}^*, \bar{\mathbf{t}}^*)) := \sum_{h_1, \dots, h_N=0}^M \Delta_N(h) \Delta_N^{(5)}(h) \prod_{i=1}^N \mu_3(h_i, \mathbf{t}^*, \bar{\mathbf{t}}^*) \quad (116)$$

$$S_4^{(1)}(\mathbf{t}_\infty, N = 2n, U(\mathbf{t}^*, \bar{\mathbf{t}}^*)) := \frac{1}{(2n)!} \sum_{h_1, \dots, h_n=0}^M \tilde{\Delta}_n^4(h) \prod_{i=1}^N e^{-2\tilde{V}(h_i, \mathbf{t}^*)} \mu_4(h_i, \mathbf{t}^*, \bar{\mathbf{t}}^*), \quad (117)$$

where

$$\mu_i(n, \mathbf{t}^*, \bar{\mathbf{t}}^*) = e^{-V(n, \mathbf{t}^*, \bar{\mathbf{t}}^*)} \mu_i(n), \quad i = 1, 3 \quad (118)$$

$$\tilde{\Delta}(h)^4 := \prod_{i < j \leq N} (h_i - h_j)^2 ((h_i - h_j)^2 - 1). \quad (119)$$

(compare with [74] where the same expression as the right-hand side was considered in the context of random partitions). Here

$$V(n, \mathbf{t}^*, \bar{\mathbf{t}}^*) = V^{(0)}(n) + \sum_{m=1}^{\infty} \left( \frac{an+b}{cn+d} \right)^m t_m^* + t_0^* \ln \left( \frac{an+b}{cn+d} \right) - \sum_{m=1}^{\infty} \left( \frac{an+b}{cn+d} \right)^{-m} \bar{t}_m^* \quad (120)$$

where  $a, b, c, d$  are arbitrary constants, conditioned by  $ad - bc \neq 0$ .

Sums (113)-(117), (122)-(125) and (126)-(129) may be viewed as discrete analogues of random matrix ensembles (compare with [32]), where eigenvalues of matrices are real non-negative numbers. Then (117) is a discrete analogue of the symplectic ensemble, (114) is a discrete analogue of orthogonal ensemble, (115) is a discrete analogue of ensemble of anti-symmetric Hermitian matrices (see Section 3.4 in [17]), and (116) is the so-called Bures ensemble, which describes random density matrices in quantum chaos problems, see [36] for the details.

From double series (42) over Frobenius coordinates of partitions we have

$$S^{(2)}(\mathbf{t}_\infty, \bar{A}, B^c) := 1 + \sum_{k=1}^{\infty} \frac{1}{(k!)^2} \sum_{\alpha_1, \dots, \alpha_k=0}^{M+1} \sum_{\beta_1, \dots, \beta_k=0}^{N+1} e^{V(\beta, \mathbf{t}^*) - V(\alpha, \mathbf{t}^*)} \frac{\bar{A}_\alpha \Delta_k(\alpha) \Delta_k(\beta) B_\beta^c}{\prod_{i,j=1}^k (\alpha_i + \beta_j + 1)} \quad (121)$$

in particular

$$S_{11}^{(2)}(\mathbf{t}_\infty, U) := 1 + \sum_{k=1} \frac{1}{(k!)^2} \sum_{\alpha_1, \dots, \alpha_k=0}^{M+1} \sum_{\beta_1, \dots, \beta_k=0}^{N+1} e^{V(\beta, \mathbf{t}^*) - V(\alpha, \mathbf{t}^*)} \frac{|\Delta_k(\alpha) \Delta_k(\beta)|}{\prod_{i,j=1}^k (\alpha_i + \beta_j + 1)}, \quad (122)$$

$$S_{33}^{(2)}(\mathbf{t}_\infty, \mathbf{t}_\infty, \mathbf{t}_\infty, \mathbf{t}^*) := 1 + \sum_{k=1} \frac{1}{(k!)^2} \sum_{\alpha_1, \dots, \alpha_k=0}^{M+1} \sum_{\beta_1, \dots, \beta_k=0}^{N+1} e^{V(\beta, \mathbf{t}^*) - V(\alpha, \mathbf{t}^*)} \frac{\Delta_k(\alpha) \Delta_k^*(\alpha) \Delta_k(\beta) \Delta_k^*(\beta)}{\prod_{i,j=1}^k (\alpha_i + \beta_j + 1)}, \quad (123)$$

$$S_{41}^{(2)}(\mathbf{t}_\infty, \mathbf{t}^*) := 1 + \quad (124)$$

$$\sum_{k=1} \frac{1}{((2k)!)^2} \sum_{\alpha_1, \dots, \alpha_k=0}^{M+1} \sum_{\beta_1, \dots, \beta_{2k}=0}^{N+1} e^{V(\beta, \mathbf{t}^*) - 2\tilde{V}(\alpha, \mathbf{t}^*)} \frac{\tilde{\Delta}_k^4(\alpha) \Delta_{2k}(\beta)}{\prod_{i=1}^k \prod_{j=1}^{2k} (\alpha_i + \beta_j + 1)(\alpha_i + \beta_j + 2)} \quad (125)$$

$$S_{44}^{(2)}(\mathbf{t}_\infty, \mathbf{t}^*) := 1 + \sum_{k=1} \frac{1}{((2k)!)^2} \sum_{\alpha_1, \dots, \alpha_k=0}^{M+1} \sum_{\beta_1, \dots, \beta_k=0}^{N+1} e^{2\tilde{V}(\beta, \mathbf{t}^*) - 2\tilde{V}(\alpha, \mathbf{t}^*)} \frac{\tilde{\Delta}_k^4(\alpha) \tilde{\Delta}_k^4(\beta)}{\prod_{i,j=1}^k (\alpha_i + \beta_j + 1)(\alpha_i + \beta_j + 2)^2(\alpha_i + \beta_j + 3)}$$

*Remark 7. Compare with sums obtained in [26]*

$$\sum_{\alpha \in \text{DP}} \Delta^*(\alpha) \prod_{i=1}^k \frac{e^{-U_{\alpha_i}(\mathbf{t}^*)}}{\alpha_i!}, \quad (126)$$

$$\sum_{\alpha \in \text{DP}} \Delta^*(\alpha)^2 \prod_{i=1}^k \frac{e^{-U_{\alpha_i}(\mathbf{t}^*)}}{(\alpha_i!)^2}, \quad (127)$$

$$\sum_{\alpha \in \text{DP}'} \tilde{\Delta}^*(\alpha)^4 \prod_{i=1}^k \frac{e^{-U_{\alpha_i}(\mathbf{t}^*) - U_{\alpha_i+1}(\mathbf{t}^*)}}{\alpha_i! (\alpha_i+1)!}, \quad (128)$$

$$\sum_{k=0}^{\infty} \frac{1}{k!} \sum_{\ell(\alpha)=\ell(\beta)=k} \Delta^*(\alpha) \Delta^*(\beta) \prod_{i=1}^k \frac{D_{\alpha_i, \beta_i}}{\alpha_i! \beta_i!} \quad (129)$$

where we remind that  $\text{DP}'$  is the set of all strict partitions  $(\alpha_1, \alpha_2, \dots, \alpha_N > 0)$  with the property  $\alpha_i > \alpha_{i+1} + 1$ ,  $i = 1, \dots, N-1$ , and

$$\tilde{\Delta}^*(\alpha)^4 := \prod_{i < j \leq N} \frac{(\alpha_i - \alpha_j)^2 ((\alpha_i - \alpha_j)^2 - 1)}{(\alpha_i + \alpha_j)^2 ((\alpha_i + \alpha_j)^2 - 1)}. \quad (130)$$

Interpretation of series (122)-(125) as discrete version of ensembles of random matrices stays unclear for us.

(B) In the same way the specialization (101) yields discrete analogues of circular ensembles in case  $q$  lies on the unit circle of the complex plane,  $q = e^{\sqrt{-1}\phi}$ ,  $\phi \neq \pi n$ ,  $n \in \mathbb{Z}$ . Here the parameterization (106)

For instance

$$S_1^{(1)}(\mathbf{t}(q), N, \mathbf{t}^*, \bar{\mathbf{t}}^*) := \frac{1}{N!} \sum_{h_1, \dots, h_N=0}^M e^{-V(q^h, \mathbf{t}^*, \bar{\mathbf{t}}^*)} |\Delta_N(q^h)| \quad (131)$$

$$S_4^{(1)}(\mathbf{t}(q), N, \mathbf{t}^*, \bar{\mathbf{t}}^*) := \frac{1}{(2n)!} \sum_{h_1, \dots, h_n=0}^M e^{-2\tilde{V}(q^h, \mathbf{t}^*, \bar{\mathbf{t}}^*)} \tilde{\Delta}_n^4(q^h), \quad N = 2n \quad (132)$$

where now

$$V(q^n, \mathbf{t}^*, \bar{\mathbf{t}}^*) = V^{(0)}(q^n) + \sum_{m=1}^{\infty} \left( \frac{aq^n + b}{cq^n + d} \right)^m t_m^* + t_0^* \ln \left( \frac{aq^n + b}{cq^n + d} \right) - \sum_{m=1}^{\infty} \left( \frac{aq^n + b}{cq^n + d} \right)^{-m} \bar{t}_m^* \quad (133)$$

*Remark 8. For  $q$  real the correspondent sums may be identified with the so-called Jackson integrals [34]*

(C) The specialization (103) where put  $x_i = e^{y_i}$  allows to rewrite (35) as

$$S_1^{(1)} = \frac{1}{\Delta_N(x)} \sum_{h_1, \dots, h_N=1}^M e^{V(h, \mathbf{t}^*)} \det(e^{y_j h_i}) \text{sgn} \Delta_N(h) \quad (134)$$

which is a discrete analogue of the following two-matrix integral

$$\int dU \int dR \det R^n \exp \left( \text{Tr} \left( UYU^\dagger R + \sum_{m \neq 0} t_m^* R^m \right) \right) \quad (135)$$

where the first integral is the integral over unitary matrices and the second is the integral over real symmetric ones,  $dU$  and  $dR$  denote the correspondent Haar measures.  $Y$  is any diagonal matrix (a source). The matrices are  $N$  by  $N$  ones. This integral may be viewed as an analogue of the Kontsevich integral.

Then with the same specialization of higher times  $\mathbf{t}$  we rewrite (45) as

$$S_{11}^{(2)} = 1 + \sum_{k=1}^{\infty} \sum_{\alpha_1, \dots, \alpha_k=0}^M \sum_{\beta_1, \dots, \beta_k=0}^N \left( \text{sgn} \Delta_k(\alpha) \left( \det \frac{1}{\alpha_i + \beta_j + 1} \right) \text{sgn} \Delta_k(\beta) \right) e^{V(\beta, \mathbf{t}^*) - V(\alpha, \mathbf{t}^*)} \quad (136)$$

Each partial sum related to a given  $k$  may be considered as a discrete version of the 3-ple integral

$$\int dU \int dA \int dB \det A^n \det B^n \det (B + UAU^\dagger)^{-k} \exp \text{Tr} \left( \sum_{m \neq 0} t_m^* A^m - \sum_{m \neq 0} t_m^* B^m \right) \quad (137)$$

where  $U$  is an unitary and both  $A$  and  $B$  are real symmetric matrices of size  $k \times k$  and where  $dU$ ,  $dA$  and  $dB$  are related Haar measure.

**New hypergeometric functions.** Now we specify factors  $e^{-U_\lambda}$  in series (35)-(38) in order to get certain generalizations of hypergeometric functions and basic hypergeometric functions.

First, we introduce the following hypergeometric series

$${}_p\Phi_r^{(\beta, N)}(\mathbf{a} + n; \mathbf{b} + n; \mathbf{t}) := \sum_{\substack{\lambda \in \mathbb{P}_\beta \\ \ell(\lambda) \leq N}} \frac{\prod_{i=1}^p (a_i + n)_\lambda}{\prod_{i=1}^r (b_i + n)_\lambda} s_\lambda(\mathbf{t}), \quad \beta = 1, 2, 4 \quad (138)$$

and their  $q$ -deformed version

$${}_p\tilde{\Phi}_r^{(\beta, N)}(\mathbf{a} + n; \mathbf{b} + n; q, \mathbf{t}) := \sum_{\substack{\lambda \in \mathbb{P}_\beta \\ \ell(\lambda) \leq N}} \frac{\prod_{i=1}^p (q^{a_i+n}; q)_\lambda}{\prod_{i=1}^r (q^{b_i+n}; q)_\lambda} s_\lambda(\mathbf{t}), \quad \beta = 1, 2, 4 \quad (139)$$

where  $\mathbb{P}_1 = \mathbb{P}$ ,  $\mathbb{P}_2 = \text{SCP}$ ,  $\mathbb{P}_4 = \text{FP}$ .

Here by  $\mathbf{a} + n$  and by  $\mathbf{b} + n$  we denote a set of parameters (“indices”)  $(a_1 + n, \dots, a_p + n)$  and  $(b_1 + n, \dots, b_r + n)$  where  $n$  and  $N$  are integer parameters and where the sum ranges over all partitions whose length (i.e. the number of non-vanishing parts) do not exceed  $N$ . The notation  $(a)_\lambda$  where  $\lambda$  has  $n$  non-vanishing parts serves for

$$(a)_\lambda := (a)_{\lambda_1} (a-1)_{\lambda_2} \cdots (a-n+1)_{\lambda_n}, \quad (a)_0 = 1 \quad (140)$$

where  $(a)_m := \frac{\Gamma(a+m)}{\Gamma(a)}$  is the so-called Pochhammer symbol. Then  $(q^a; q)_\lambda$  is the  $q$ -deformed version of  $(a)_\lambda$

$$(q^a; q)_\lambda := (q^a; q)_{\lambda_1} \cdots (q^{a-n+1}; q)_{\lambda_n} \quad (141)$$

defined via the  $q$ -deformed Pochhammer’s symbols:

$$(q^a; q)_0 := 1, \quad (q^a; q)_n := (1 - q^a) \cdots (1 - q^{a+n-1}) \quad (142)$$

*Remark 9.* In case  $t_m = \frac{1}{m} \sum_{i=1}^L x_i^m$  the summation range is restricted by the condition  $\ell(\lambda) \leq L$  because the Schur functions vanish on the partitions whose length exceed  $L$ .

*Remark 10.* As one may notice that by specialization of  $\mathbf{t} = (t, 0, 0, \dots)$  and  $N = 1$  we obtain the generalized hypergeometric function [38]

$${}_p\Phi_r^{(1,1)}(\mathbf{a} + n; \mathbf{b} + n; \mathbf{t}) = \sum_{k=0}^{\infty} \frac{\prod_{i=1}^p (a_i + n)_k}{\prod_{i=1}^r (b_i + n)_k} \frac{t^k}{k!}$$

Taking  $\mathbf{t} = \left(\frac{x}{1}, \frac{x^2}{2}, \frac{x^3}{3}, \dots\right)$  we obtain

$${}_p\Phi_r^{(1,1)}(\mathbf{a} + n; \mathbf{b} + n; \mathbf{t}) = \sum_{k=0}^{\infty} \frac{\prod_{i=1}^p (a_i + n)_k}{\prod_{i=1}^r (b_i + n)_k} x^k$$

$${}_p\tilde{\Phi}_r^{(1,1)}(\mathbf{a} + n; \mathbf{b} + n; q, \mathbf{t}) := \sum_{k=0}^{\infty} \frac{\prod_{i=1}^p (q^{a_i+n}; q)_k}{\prod_{i=1}^r (q^{b_i+n}; q)_k} x^k$$

which, say, for  $b_1 + n = 1$  yields respectively the known generalized and the basic hypergeometric functions [38].

The hypergeometric function (138) is obtained as a specific case of (35) if we choose  $U$  variables as follows

$$U_m = \sum_{i=1}^q \log(b_i)_m - \sum_{i=1}^p \log(a_i)_m \quad (143)$$

The choice

$$U_m = \sum_{i=1}^r \log(q^{b_i}; q)_m - \sum_{i=1}^p \log(q^{a_i}; q)_m \quad (144)$$

gives rise to (138).

**Hypergeometric functions II** We also introduce

$${}_p\Psi_r^{(4,N)}(\mathbf{a} + n; \mathbf{b} + n; \mathbf{t}) := \sum_{\substack{\lambda \in \mathbb{P} \\ \ell(\lambda) \leq N}} \frac{\prod_{i=1}^p (a_i + n)_{\lambda,4}}{\prod_{i=1}^r (b_i + n)_{\lambda,4}} s_{\lambda \cup \lambda}(\mathbf{t}) \quad (145)$$

$${}_p\Psi_r^{(1,N)}(\mathbf{a} + n; \mathbf{b} + n; \mathbf{t}) := \sum_{\substack{\lambda \in \mathbb{P} ev \\ \ell(\lambda) \leq N}} \frac{\prod_{i=1}^p (a_i + n)_{\lambda,1}}{\prod_{i=1}^r (b_i + n)_{\lambda,1}} s_{\lambda}(\mathbf{t}) \quad (146)$$

and their  $q$ -deformed version

$${}_p\tilde{\Psi}_r^{(4,N)}(\mathbf{a} + n; \mathbf{b} + n; q, \mathbf{t}) := \sum_{\substack{\lambda \in \mathbb{P} \\ \ell(\lambda) \leq N}} \frac{\prod_{i=1}^p (q^{a_i+n}; q)_{\lambda,4}}{\prod_{i=1}^r (q^{b_i+n}; q)_{\lambda,4}} s_{\lambda \cup \lambda}(\mathbf{t}) \quad (147)$$

$${}_p\tilde{\Psi}_r^{(1,N)}(\mathbf{a} + n; \mathbf{b} + n; q, \mathbf{t}) := \sum_{\substack{\lambda \in \mathbb{P} ev \\ \ell(\lambda) \leq N}} \frac{\prod_{i=1}^p (q^{a_i+n}; q)_{\lambda,1}}{\prod_{i=1}^r (q^{b_i+n}; q)_{\lambda,1}} s_{\lambda}(\mathbf{t}) \quad (148)$$

where

$$(a)_{\lambda, \beta} := (a)_{\lambda_1} (a - \frac{1}{2}\beta)_{\lambda_2} \cdots (a - \frac{1}{2}(n-1)\beta)_{\lambda_n}, \quad (a)_{0,4} = 1 \quad (149)$$

Thanks to the formula  $s_{\lambda}(-\mathbf{t}) = (-1)^{|\lambda|} s_{\lambda^{tr}}(\mathbf{t})$  we have

$${}_p\Psi_r^{(1,N)}(\mathbf{a} + n; \mathbf{b} + n; q, \mathbf{t}) = {}_p\Psi_r^{(4,N)}(\mathbf{a} + n; \mathbf{b} + n; q, -\mathbf{t}) \quad (150)$$

$${}_p\tilde{\Psi}_r^{(1,N)}(\mathbf{a} + n; \mathbf{b} + n; q, \mathbf{t}) = {}_p\tilde{\Psi}_r^{(4,N)}(\mathbf{a} + n; \mathbf{b} + n; q, -\mathbf{t}) \quad (151)$$

*Remark 11.* Having in mind the known relation (see [18], [39])

$$\int_{U \in U(N, \mathbb{F}_{\beta})} s_{2\lambda}(XU) dU = \frac{J_{\lambda}^{(\frac{2}{\beta})}(XX^{\dagger})}{J_{\lambda}^{(\frac{2}{\beta})}(\mathbf{1}_N)}$$



we obtain

$$\int_{U \in \mathbb{U}(2k, \mathbb{F}_\beta)} {}_p\Psi_r^{(\beta, N)}(\mathbf{a} + n; \mathbf{b} + n; XU) dU := \sum_{\substack{\lambda \in \mathbb{P} \\ \ell(\lambda) \leq k}} \frac{\prod_{i=1}^p (a_i + n)_{\lambda, \beta}}{\prod_{i=1}^r (b_i + n)_{\lambda, \beta}} \frac{J_{\lambda}^{(\frac{2}{\beta})}(XX^\dagger)}{J_{\lambda}^{(\frac{2}{\beta})}(\mathbf{1}_N)}, \quad \beta = 1, 2, 4 \quad (152)$$

and their  $q$ -deformed version

$$\int_{U \in \mathbb{U}(2k, \mathbb{F}_\beta)} {}_p\tilde{\Psi}_r^{(\beta, N)}(\mathbf{a} + n; \mathbf{b} + n; q, XU) dU := \sum_{\substack{\lambda \in \mathbb{P} \\ \ell(\lambda) \leq k}} \frac{\prod_{i=1}^p (q^{a_i+n}; q)_{\lambda, \beta}}{\prod_{i=1}^r (q^{b_i+n}; q)_{\lambda, \beta}} \frac{J_{\lambda}^{(\frac{2}{\beta})}(XX^\dagger)}{J_{\lambda}^{(\frac{2}{\beta})}(\mathbf{1}_N)}, \quad \beta = 1, 2, 4 \quad (153)$$

Thanks to results of Subsection 2.2 we have Pfaffian representation for each of the introduced hypergeometric functions. In particular

$${}_p\Phi_r^{(4, N)}(\mathbf{a} + n; \mathbf{b} + n; \mathbf{t}) = \frac{1}{\Delta_N(x)} \text{Pf}(x_j - x_i)_p \Phi_r^{(4, N)}(\mathbf{a} + n; \mathbf{b} + n; x_i x_j) \quad (154)$$

$${}_p\Psi_r^{(4, N)}(\mathbf{a} + n; \mathbf{b} + n; \mathbf{t}) = \frac{1}{\Delta_N(x)} \text{Pf}(x_j - x_i)_p \Phi_r^{(4, N)}(\mathbf{a} + n; \mathbf{b} + n; x_i x_j) \quad (155)$$

*Remark 12.* Let us note that the hypergeometric series (138), (145), (145) are different from the so-called (case  $\mathbb{C}$ ) hypergeometric function of matrix argument [35], [34]

$$\sum_{\substack{\lambda \in \mathbb{P} \\ \ell(\lambda) \leq N}} \frac{\prod_{i=1}^p (q^{a_i}; q)_\lambda}{\prod_{i=1}^r (q^{b_i}; q)_\lambda} \frac{s_\lambda(x^{(N)})}{H_\lambda}$$

and hypergeometric series (139) are different from Milne's hypergeometric series [33], [34]

$$\sum_{\substack{\lambda \in \mathbb{P} \\ \ell(\lambda) \leq N}} \frac{\prod_{i=1}^p (q^{a_i}; q)_\lambda}{\prod_{i=1}^r (q^{b_i}; q)_\lambda} \frac{s_\lambda(x^{(N)})}{H_\lambda(q)}$$

which are examples of KP tau functions [29], [30]. In these formulas  $H_\lambda$  and  $H_\lambda(q)$  are the hook-product and the  $q$ -deformed hook product respectively.

### 3 Fermionic representation

We suppose that the reader is familiar with the definition of the Fermi operators and the vacuum expectation value, for notations see Appendix B.

One may prove the following relations

$$S^{(1)}(\mathbf{t}, N, U, \bar{A}) = \langle N | \Gamma(\mathbf{t}) \mathbb{T}(U) g^{--}(\bar{A}) | 0 \rangle \quad (156)$$

where

$$g^{--}(\bar{A}) = g^{--}(A, a) = e^{\frac{1}{2} \sum_{n, m \in \mathbb{Z}} A_{nm} \psi_n \psi_m + \sum_{n \in \mathbb{Z}} a_n \psi_n \phi_0} \quad (157)$$

In particular we have

$$S_0^{(1)}(\mathbf{t}, N; M, U) := \langle N | \Gamma(\mathbf{t}) e^{\sum_{M \geq m > n} \psi_m \psi_n + \sum_{m \leq M} \psi_m \phi_0} | 0 \rangle \quad (158)$$

$$S_1^{(1)}(\mathbf{t}; N, U) := \langle N | \Gamma(\mathbf{t}) \mathbb{T}(U) e^{\sum_{m > n} \psi_m \psi_n + \sum_{m \in \mathbb{Z}} \psi_m \phi_0} | 0 \rangle \quad (159)$$

$$S_2^{(1)}(\mathbf{t}, N = 2n; U, c) := \langle N | \Gamma(\mathbf{t}) \mathbb{T}(U) e^{\sum_{m < n} \psi_{2c-m+1} \psi_m} | 0 \rangle \quad (160)$$

$$S_3^{(1)}(\mathbf{t}, N, \mathbf{t}'; U) := \langle N | \Gamma(\mathbf{t}) \mathbb{T}(U) e^{\sum_{m > n} Q_{(\alpha(n), \alpha(m))}(\frac{1}{2} \mathbf{t}') \psi_m \psi_n + \sum_{m \in \mathbb{Z}} Q_{(\alpha(m))} \psi_m \phi_0} | 0 \rangle \quad (161)$$

$$S_4^{(1)}(\mathbf{t}, N = 2n; U) := \langle N | \Gamma(\mathbf{t}) \mathbb{T}(U) e^{\sum_{m \in \mathbb{Z}} \psi_m \psi_{m-1}} | 0 \rangle \quad (162)$$

$$S_5^{(1)}(\mathbf{t}, N; U, f) := \langle N | \Gamma(\mathbf{t}) \mathbb{T}(U) e^{\sum_{m > n} \frac{f(m) - f(n)}{f(m) + f(n)} \psi_m \psi_n + \sum_{m \in \mathbb{Z}} \psi_m \phi_0} | 0 \rangle \quad (163)$$

$$S_6^{(1)}(\mathbf{t}, N; U, f) := \langle N | \Gamma(\mathbf{t}) \mathbb{T}(U) e^{\sum_{m > n} \frac{f(m) - f(n)}{1 - f(m)f(n)} \psi_m \psi_n + \sum_{m \in \mathbb{Z}} \psi_m \phi_0} | 0 \rangle \quad (164)$$

$$S_7^{(1)}(\mathbf{t}, N; U, f) := \langle N | \Gamma(\mathbf{t}) \mathbb{T}(U) e^{\sum_{m > n} \frac{f(m) - f(n)}{(f(m) + f(n))^2} \psi_m \psi_n + \sum_{m \in \mathbb{Z}} \psi_m \phi_0} | 0 \rangle \quad (165)$$

where

$$\Gamma(\mathbf{t}) = e^{J(\mathbf{t})}, \quad \bar{\Gamma}(\bar{\mathbf{t}}) = e^{\bar{J}(\bar{\mathbf{t}})} \quad (166)$$

$$J(\mathbf{t}) := \sum_{n=1}^{\infty} J_n t_n, \quad \bar{J}(\bar{\mathbf{t}}) := \sum_{n=1}^{\infty} J_{-n} \bar{t}_n, \quad J_n := \sum_{m \in \mathbb{Z}} \psi_m \psi_{m+n}^\dagger \quad (167)$$

and

$$\mathbb{T}(U) := \exp \sum_{i < 0} U_i \psi_i \psi_i^\dagger - \sum_{i \geq 0} U_i \psi_i \psi_i^\dagger \quad (168)$$

where the fermionic operators are defined as in [6], see Appendix B

*Remark 13.* Pfaffian representation presented in Subsection 2.2 considered above may be obtained from the Wick's rule.

*Remark 14.* One can write

$$g^{--}(\bar{A}_1) := e^{\sum_{m > n} \psi_m \psi_n + \sum_{m \in \mathbb{Z}} \psi_m \phi_0} = e^{\oint \psi(x^{-1})(\psi(x) + \phi_0) \frac{1}{1-x} dx} \quad (169)$$

$$g^{--}(\bar{A}_2) := e^{\sum_{m < n} (-1)^m \psi_{2c-m+1} \psi_m} = e^{\oint x^{-2c-2} \psi(x) \psi(-x) dx} \quad (170)$$

$$g^{--}(\bar{A}_4) := e^{\sum_{m \in \mathbb{Z}} \psi_m \psi_{m-1}} = e^{\oint \psi(x^{-1}) \psi(x) dx} \quad (171)$$

The corollary of the right hand side expressions is the fact that sums (35), (36) and (38) may be re-written as certain multiply integrals ( $\frac{1}{2}N$ -ply integrals for  $S_2^{(1)}$ ,  $S_4^{(1)}$ , and  $N$ -ply integrals for  $S_1^{(1)}$ ), this will be considered in details in [1]. Now, we shall mention a general remark.

Imagine that a sum (9) we can present  $A_{nm}$  as moments, or, the same we can solve the following inverse moment problem: given  $A_{nm} = -A_{mn}$ ,  $m, n \geq 0$  to find such an integration domain  $D$  and an antisymmetric measure  $dA(x, y) = -dA(y, x)$  such that

$$A_{nm} = \int_D x^n y^m dA(x, y), \quad n, m \geq 0 \quad (172)$$

Also

$$a_n = \int_\gamma x^n da(x) \quad (173)$$

If we have (172) then in case  $N = 2n$  we can write  $N$ -ply integral

$$S^{(1)}(\mathbf{t}, 2n, U, \bar{A}) = e^{-\sum_{i=0}^{N-1} U_i} \int_{D^n} \left( \prod_{i=1}^{2n} e^{\xi_r(\mathbf{t}, x_i)} \cdot \Delta_{2n}(x) \right) \text{Pf} [dA(x_i, x_j)] \quad (174)$$

where  $\xi_r(\mathbf{t}, x)$  is the following  $\Psi$ DO operator

$$\xi_r(\mathbf{t}, x) = \sum_{m=1}^{\infty} t_m (xr(D))^m, \quad D = x\partial_x \quad (175)$$

and  $r$  is related to  $U$  as follows

$$r(n) = e^{U_n - U_{n+1}} \quad (176)$$

The case  $U = 0$  causes  $\xi_r(\mathbf{t}, x) = \sum_{m=1}^{\infty} t_m x^m$  and we obtain more familiar expression

$$S^{(1)}(\mathbf{t}, 2n, U = 0, \bar{A}) = \int_{D^n} \prod_{i=1}^{2n} e^{\sum_{m=1}^{\infty} t_m x_i^m} \Delta_{2n}(x) \text{Pf} [dA(x_i, x_j)] \quad (177)$$

In case  $N = 2n + 1$  we have more involved expressions which will be written down in a more detailed version.

In case the solution of the inverse problem is not unique we have a set of different integral representations for the sum (9).

### Other representations :

For certain sums like (161) we present a different representation as follows

$$S_3^{(1)}(\mathbf{t}, N, \mathbf{t}'; U) := \langle N | \Gamma(\mathbf{t}) \Gamma_B(\mathbf{t}') \mathbb{T}(U) e^{\sum_{m > n} \phi_{\alpha(n)} \phi_{\alpha(m)} \psi_m \psi_n + \sum_{m \in \mathbb{Z}} Q_{(\alpha(m))} \psi_m \phi_0} | 0 \rangle \quad (178)$$

**Fermionic representation for  $S^{(2)}$ .** For  $S^{(2)}$  of (42) we have a similar relations

$$S^{(2)}(\mathbf{t}; U, \bar{A}, \bar{B}) = \langle 0 | \Gamma(\mathbf{t}) \mathbb{T}(U) g^+(\bar{B}) g^-(\bar{A}) | 0 \rangle \quad (179)$$

where  $\mathbb{T}(U)$  may found in (168)

$$g^-(\bar{A}) = g^-(A, a) = e^{\frac{1}{2} \sum_{n,m \geq 0} A_{nm} \psi_n \psi_m + \sum_{n > 0} a_n \psi_n \phi_0} \quad (180)$$

$$g^+(\bar{B}) = g^+(B, b) = e^{\frac{1}{2} \sum_{n,m \geq 0} B_{nm} (-1)^{n+m} \psi_{-n-1}^\dagger \psi_{-m-1}^\dagger + \sum_{n > 0} (-1)^n b_n \phi_0 \psi_{-n-1}^\dagger} \quad (181)$$

For those who are familiar with [13] these fermionic relations yields a direct proof of Propositions 1 and 2. In the next section we will consider it in more details.

## 4 Small and Large BKP tau functions

### 4.1 A class of "small" BKP (sBKP) tau functions

We start with the "small" BKP case because it is more simple and illustrative. Concerning this case see [26].

A general tau function of the sBKP hierarchy may be written as

$$\tau^{sBKP}(\mathbf{t}') = \langle 0 | \Gamma_B(\mathbf{t}') e^{\sum_{n,m \in \mathbb{Z}} A_{nm} \phi_n \phi_m} | 0 \rangle \quad (182)$$

and a tau function of 2-sBKP hierarchy as

$$\tau^{sBKP}(\mathbf{t}', \bar{\mathbf{t}}') = \langle 0 | \Gamma_B(\mathbf{t}') e^{\sum_{n,m \in \mathbb{Z}} A_{nm} \phi_n \phi_m} \Gamma_B(\bar{\mathbf{t}}') | 0 \rangle \quad (183)$$

The sBKP hierarchy is obtained from the KP hierarchy by a reduction. In the sBKP reduction, even times are set equal to zero and we shall mark it by " ' " :  $\mathbf{t}' = (t'_1, 0, t'_3, 0, t'_5, \dots)$ , then

$$\Gamma_B(\mathbf{t}') = \exp \sum_{n \geq 1, \text{ odd}} H_n^B t'_n, \quad \bar{\Gamma}_B(\bar{\mathbf{t}}') = \exp \sum_{n \geq 1, \text{ odd}} H_{-n}^B \bar{t}'_n \quad (184)$$

where

$$H_n^B = \frac{1}{2} \sum_{i \in \mathbb{Z}} (-1)^{i+1} \phi_i \phi_{-i-n}. \quad (185)$$

The sets of parameters  $\mathbf{t}'$  and  $\bar{\mathbf{t}}'$  are called sBKP higher times.

**"Easy" tau functions** We shall consider a simple case, where all terms in the sum of the exponent (182) commute, namely, tau functions

$$\tau^{sBKP}(\mathbf{t}', U, A) = \langle 0 | \Gamma_B(\mathbf{t}') \mathbb{T}_B(U) e^{\sum_{n > m > 0} A_{nm} \phi_n \phi_m + \sum_{n > 0} a_n \phi_n \phi_0 \sqrt{2}} | 0 \rangle \quad (186)$$

where

$$\mathbb{T}_B(U) = \exp \left( - \sum_{n > 0} (-1)^{n+1} U_n \phi_n \phi_{-n} \right) \quad (187)$$

We want to single out the  $U$ -dependence though it may be included into the redefinition of  $A$  as  $A_{nm} \rightarrow e^{-U_m - U_n} A_{nm}$ .

We shall refer tau functions (186) as easy sBKP tau functions.

We have

**Proposition 12.**

$$\tau^{sBKP}(\mathbf{t}', U, A) = \sum_{\alpha \in \text{DP}} e^{-U_{\{\alpha\}}} A_\alpha Q_\alpha(\mathbf{t}') \quad (188)$$

where the sum ranges over all strict partitions  $\alpha = (\alpha_1, \dots, \alpha_k)$ ,  $k = 0, 1, 2, \dots$ , where

$$U_{\{\alpha\}} = \sum_{i=1}^k U_{\alpha_i} \quad (189)$$

and where  $A_{\{\alpha\}}$  is the Pfaffian of the  $k \times k$  antisymmetric matrix  $\tilde{A}$  defined as follows:  
for even  $k$  its entries are

$$\tilde{A}_{nm} = A_{\alpha_n \alpha_m}, \quad n, m = 1, \dots, k; \quad (190)$$

for  $k$  odd we take

$$\tilde{A}_{nm} = A_{\alpha_n \alpha_m}, \quad n, m = 1, \dots, k; \quad \tilde{A}_{n, k+1} = -\tilde{A}_{k+1, n} = a_n, \quad n = 1, \dots, k+1 \quad (191)$$

It is assumed that for  $\alpha = 0$   $A_{\{0\}} = Q_0 = e^{-U_{\{0\}}} = 1$ .

For proof we notice that  $e^{\sum_{n>0} a_n \phi_n \phi_0 \sqrt{2}} = 1 + \sum_{n>0} a_n \phi_n \phi_0 \sqrt{2}$ , and take into account that

$$\mathbb{T}_B(U) \phi_{\alpha_1} \cdots \phi_{\alpha_k} |0\rangle = e^{-U_{\{\alpha\}}} \phi_{\alpha_1} \cdots \phi_{\alpha_k} |0\rangle$$

Then we obtain (188) thanks to Lemma 2 in Appendix E.3.

**Example 1.** Choosing

$$A_{nm} = Q_{(n,m)}(\tfrac{1}{2}\bar{\mathbf{t}}'), \quad n > m > 0, \quad a_n = Q_{(n)}(\tfrac{1}{2}\bar{\mathbf{t}}')$$

where  $Q_{(n,m)}$  is the projective Schur function related to a partition  $(n, m)$  and  $\bar{\mathbf{t}}'$  are parameters we obtain

$$\tau^{sBKP}(\mathbf{t}', U, A) = \sum_{\alpha \in \text{DP}} e^{-U_{\{\alpha\}}} Q_{\alpha}(\mathbf{t}') Q_{\alpha}(\bar{\mathbf{t}}') \quad (192)$$

which is actually an example of a tau function (183), see [31].

**Example 2.** Choosing

$$A_{nm} = 1, \quad n > m, \quad a_n = 1$$

we obtain

$$\tau^{sBKP}(\mathbf{t}', U) = \sum_{\alpha \in \text{DP}} e^{-U_{\{\alpha\}}} Q_{\alpha}(\mathbf{t}') \quad (193)$$

The right-hand side of (193) appeared in [69] as a generating function for partition functions related to oscillating strict partitions.

*Remark 15.* Summation range  $\infty > \alpha_1 > \cdots > \alpha_k > 0$ ,  $k = 0, 1, 2$  in sums over all strict partitions may be replaced by sets of strict partitions whose parts may take values in a given set of natural numbers which we can write as a strict partition, say,  $\gamma = (\gamma_1, \dots, \gamma_N)$ ,  $\gamma_1 > \cdots > \gamma_N$ ,  $N$  may be infinite. This may be obtained by equating  $e^{-U_n}$  to zero in case  $n$  is not equal to any of  $\gamma_i$ . We obtain the following sBKP tau function

$$\tau_{\gamma}^{sBKP}(\mathbf{t}', U) := \sum_{\substack{\alpha \in \text{DP} \\ \alpha \subseteq \gamma}} e^{-U_{\{\alpha\}}} Q_{\alpha}(\mathbf{t}') \quad (194)$$

**Relation to solitons.** Let us note that for  $\mathbf{t}' = (1, 0, 0, 0, \dots) =: \mathbf{t}_{\infty}$  and

$$U_n = U_n(\mathbf{t}^*, \bar{\mathbf{t}}^*) = U_n^{(0)} - \log n! - \sum_{m=1,3,\dots} n^m t_m^* + \sum_{m=1,3,\dots} n^{-m} \bar{t}_m^*, \quad n > 0$$

thanks to Lemma 5 in Appendix E.4 the right-hand side of (193) gives rise to a multisoliton 2-sBKP tau function where  $\mathbf{t}^*$  and  $\bar{\mathbf{t}}^*$  play the role of higher times. Indeed, with the help of (366) the right-hand side of (193) reads as

$$\begin{aligned} & \langle 0 | \Gamma_B(\mathbf{t}_{\infty}) \mathbb{T}_B(U(\mathbf{t}^*, \bar{\mathbf{t}}^*)) e^{\sum_{n>m>0} \phi_n \phi_m + \sum_{n>0} \phi_n \phi_0 \sqrt{2}} | 0 \rangle = \\ &= 1 + \sum_{k=1}^{\infty} \sum_{\alpha_1 > \cdots > \alpha_k > 0} \prod_{i=1}^k e^{\sum_{m=1,3,\dots} (\alpha_i^m t_m^* - \alpha_i^{-m} \bar{t}_m^* - U_{\alpha_i}^{(0)})} \prod_{i < j} \frac{\alpha_i - \alpha_j}{\alpha_i + \alpha_j} = \\ &= \langle 0 | \Gamma_B(\mathbf{t}^*) e^{\sum_{n>m>0} e^{-U_m^{(0)} - U_n^{(0)}} \phi(n) \phi(m)} e^{\sum_{n>0} e^{-U_n^{(0)}} \phi(n) \phi_0 \sqrt{2}} \Gamma_B(\bar{\mathbf{t}}^*) | 0 \rangle \end{aligned} \quad (195)$$

where

$$\phi(z) := \sum_{n \in \mathbb{Z}} z^n \phi_n$$

*Remark 16.* On soliton solutions of integrable equations see [43], on multisolitons of sBKP hierarchy see [6]. The constant  $U_n^{(0)}$  plays the role of the initial phase which defines the initial location of the soliton marked by  $n$ . One can "remove" any soliton by 'sending it to infinity', i.e. via  $e^{-U_n^{(0)}} \rightarrow 0$ . See Remark 15, where  $\gamma$  may be interpreted as the soliton momentums in the  $N$ -soliton solution to sBKP hierarchy.

## 5 The "large" BKP (lBKP) and "large" 2-BKP tau functions

A general large BKP and large 2-BKP tau functions may be expressed as the following fermionic expectation value, respectively :

$$\tau_N(l, \mathbf{t}) = \langle N + l | \Gamma(\mathbf{t}) g | l \rangle \quad (196)$$

and

$$\tau_N(l, \mathbf{t}, \bar{\mathbf{t}}) = \langle N + l | \Gamma(\mathbf{t}) g \bar{\Gamma}(\bar{\mathbf{t}}) | l \rangle \quad (197)$$

where  $N$  is an integer where  $\Gamma(\mathbf{t})$  is the same as in (166) and where

$$g = \exp \sum_{n,m} A_{nm} \psi_n \psi_m + B_{nm} \psi_n^\dagger \psi_m^\dagger + D_{nm} \psi_n \psi_m^\dagger + \sum_n (a_n \psi_n + b_n \psi_n^\dagger) \phi_0 \quad (198)$$

Here  $A_{nm} = -A_{mn}$  and  $B_{nm} = -B_{mn}$ . It is due to the presence of  $\phi_0$  and thanks to equations (333), (331) the number  $N$  may be odd as well as even.

Parameters  $\mathbf{t} = (t_1, t_2, \dots)$  and  $\bar{\mathbf{t}} = (\bar{t}_1, \bar{t}_2, \dots)$  are called higher times of the 2-BKP hierarchy.

The large BKP tau function (196) was introduced in [13], it solves large BKP Hirota equation written down in [13]. Hirota equations for 2-lBKP are written down in the Appendix.

At the present paper lBKP tau functions (196) are mainly used to study multiple sums, while 2-lBKP tau functions (197) will be used to study multiple integrals in [1].

**Easy tau functions.** If for  $g$  we choose any product of three special  $g$  which are

$$g = g^-(\bar{A}) := \left( \exp \sum_{n,m \geq L} A_{nm} \psi_n \psi_m + \sum_{n \geq L} a_n \psi_n \phi_0 \right), \quad (199)$$

$$g = g^+(\bar{B}) := \left( \exp \sum_{n,m \geq L} B_{nm} \psi_{-n-1}^\dagger \psi_{-m-1}^\dagger + \sum_{n \geq L} b_n \phi_0 \psi_{-n-1}^\dagger \right) \quad (200)$$

where  $l$  is the right hand charge. We will also use

$$g = g^{-(D)} := \left( \exp \sum_{n,m \geq L} D_{nm} \psi_n \psi_{-m-1}^\dagger \right) \quad (201)$$

we obtain simply expressions for (196) and (197). This is because of the fact that all fermionic operators in the exponents in (199),(200),(201) anticommute. (In this sense we treat fermions as Grassmannian variables).

The integer  $L$  indicating the summation range is arbitrary. Next we consider basic examples.

First, let us consider

$$\tau = \langle l | \Gamma(\mathbf{t}) \mathbb{T}(U) g^{-(D)} | l \rangle = \sum \det[D_{h_i, h_j}] e^{-U_{\{h\}} s_{\{h\}}}(\mathbf{t}) \quad (202)$$

then choosing  $D_{nm} = s_{(n|m)}(\bar{\mathbf{t}})$  we obtain that it is equal to

$$c_l \sum_{\lambda \in \mathbb{P}} e^{-U_\lambda(l)} s_\lambda(\mathbf{t}) s_\lambda(\bar{\mathbf{t}}) \quad (203)$$

where  $U_\lambda(l)$  and  $c_l$  are the same as written down below in Proposition 13. This is the well-known solution of TL hierarchy where the set  $l, \mathbf{t}, \bar{\mathbf{t}}$  plays the role of higher times. These series were called hypergeometric

tau functions in [29] because they keep many properties of ordinary generalized hypergeometric functions (where the role of Gauss equation takes the so-called string equation). Various specifications of this tau function were widely used in various problems: in analyze of generalized Kontsevich model [40], in 2D chromodynamics [48], [49], [86], some  $c = 1$  string theory calculations [46], evaluation of Hurwitz numbers [51], generalized hypergeometric functions [32] (where the general form (203) was studied), for models of random partitions [73], for perturbation series in coupling constants for two-matrix and for normal matrix models [21], [32], for construction of solvable matrix integrals [97], for some calculus in Seiberg-Witten theory [47], Gromov-Witten theory [53], [54], physics of electronic liquid [56], models of random turn motion by M.Fisher [21], so-called melting crystals problem [28], [61] 6-vertex model [62], [63], [64], [65], and in many others problems.

We hope that the relatives of this series which will be presented below will also find wide applications.

### 5.1 IBKP tau functions $\tau_N(l, \mathbf{t}, U, \bar{A})$

We will consider

$$\tau_N(l, \mathbf{t}, U, \bar{A}) := \langle N + l | \Gamma(\mathbf{t}) \mathbb{T}(U) g^{--}(A, a) | l \rangle, \quad (204)$$

where

$$g^{--}(A, a) := e^{\frac{1}{2} \sum_{n,m} A_{nm} \psi_m \psi_n + \sum_{n \in \mathbb{Z}} a_n \psi_n \phi_0 \sqrt{2}} \quad (205)$$

and  $\mathbb{T}(U)$  is as in (168). Tau function (204) vanishes if  $N < 0$ . We remind that we deal with a pair  $\bar{A} = (A, a)$  which consists of an infinite antisymmetric matrix  $A$  and an infinite vector  $a$ .

We have the following (compare with Proposition 1)

**Proposition 13.**

$$\tau_N(l, \mathbf{t}, U, \bar{A}) = c_l \sum_{\substack{\lambda \\ \ell(\lambda) \leq N}} e^{-U_\lambda(l)} \bar{A}_{\{h\}}(l) s_\lambda(\mathbf{t}) \quad (206)$$

where

$$U_\lambda(l) := \sum_{i=1}^N U_{\lambda_i - i + N + l} \quad (207)$$

and  $\bar{A}_{\{h\}}(l)$  is the Pfaffian of a matrix  $\tilde{A}$  defined as the Pfaffian of an antisymmetric  $2n \times 2n$  matrix  $\tilde{A}$  as follows:

$$\bar{A}_h(l) := \text{Pf}[\tilde{A}] \quad (208)$$

where for  $N = 2n$  even

$$\tilde{A}_{ij} = -\tilde{A}_{ji} := A_{h_i+l, h_j+l}, \quad 1 \leq i < j \leq 2n \quad (209)$$

and for  $N = 2n - 1$  odd

$$\tilde{A}_{ij} = -\tilde{A}_{ji} := \begin{cases} A_{h_i+l, h_j+l} & \text{if } 1 \leq i < j \leq 2n-1 \\ a_{h_i+l} & \text{if } 1 \leq i < j = 2n. \end{cases} \quad (210)$$

We set  $\bar{A}_0(l) = 1$ .

The constant  $c_l$  is defined by

$$c_l = \begin{cases} e^{-U_{l-1} - \dots - U_0} & l > 0 \\ 1 & l = 0 \\ e^{U_l + \dots + U_{-1}} & l < 0 \end{cases} \quad (211)$$

As we see  $\bar{A}_h(0)$  and  $U_\lambda(0)$  coincides respectively with  $\bar{A}_h$  and  $U_\lambda$  defined in Section 2. This proves Proposition 1.

*Remark 17.* The right-hand side of (206) may be also obtained as a IDKP tau function.

*Remark 18.* The right-hand side of (206) may be also obtained as a special limit of a tau function (225) below. This is a case where  $e^{-U_\lambda}$  vanishes if the length of partition  $\lambda$  exceeds  $N$ , to ensure this we put  $e^{-U_{-N-1}} = 0$ , or, the same, we put  $U_{-N-1} = +\infty$ .

As examples we have

$$\tau_N^{lBKP-sBKP}(l, \mathbf{t}, \mathbf{t}', U) := \sum_{\substack{\lambda \\ \ell(\lambda) \leq N}} e^{-U\lambda(l)} Q_{l+\lambda^-} \left( \frac{1}{2} \mathbf{t}' \right) s_\lambda(\mathbf{t}) \quad (212)$$

$$\tau_N^{lBKP-sBKP}(l, \mathbf{t}, \mathbf{t}', U=0) := \sum_{\substack{\lambda \\ \ell(\lambda) \leq N}} Q_{l+\lambda^-} \left( \frac{1}{2} \mathbf{t}' \right) s_\lambda(\mathbf{t}) \quad (213)$$

$$\tau_N(l, \mathbf{t}, \mathbf{t}', U) := \sum_{\substack{\lambda \\ \ell(\lambda) \leq N}} e^{-U\lambda(l)} s_\lambda(\mathbf{t}) \quad (214)$$

$$\tau_N(\mathbf{t}) := \sum_{\substack{\lambda \\ \ell(\lambda) \leq N}} s_\lambda(\mathbf{t}) \quad (215)$$

where  $l + \lambda^-$  denotes the strict partition whose  $i$ -th part is equal to  $\lambda_i - i + N + l$ . Here  $\mathbf{t}'$  is the set of variables denoted  $(t'_1, t'_3, t'_5, \dots)$  and  $Q_\alpha(\frac{1}{2} \mathbf{t}')$  is the projective Schur functions [18] related to a strict partition  $\alpha$ . As we shall show later IBKP tau functions (212) and (213) are also sBKP tau function whose higher times are  $\mathbf{t}'$ .

## 5.2 IBKP tau functions $\tau_N(l, \mathbf{t}, U, \bar{A}, \bar{B})$

We begin with a rather special  $O(2\infty + 1)$  element as follows

$$g_o = g_o^+ g_o^-, \quad (216)$$

where

$$g_o^- = e^{\sum_{n>m \geq 0} \psi_n \psi_m + \sum_{n \geq 0} \psi_n \phi_0}, \quad g_o^+ = e^{\sum_{n>m \geq 0} (-)^{n+m} \psi_{-m-1}^\dagger \psi_{-n-1}^\dagger + \sum_{n \geq 0} \phi_0 \psi_{-n-1}^\dagger} \quad (217)$$

which are exponentials of nilpotents and mutually commuting operators in the Fock space.

Let us recall that the Fock space  $\mathcal{F}$  admits a decomposition as an orthogonal direct sum of the subspaces  $\mathcal{F}_N$  of states with charge  $N$

$$\mathcal{F} = \oplus_{N \in \mathbb{Z}} \mathcal{F}_N. \quad (218)$$

We have

$$g_o|0\rangle = |\Omega\rangle, \quad |\Omega\rangle = \sum_{N \in \mathbb{Z}} |\Omega_N\rangle \quad (219)$$

where each vector  $|\Omega_N\rangle$  belongs to the subspace  $\mathcal{F}_N$ . The result we need is

**Lemma 1.** *The vector  $|\Omega_0\rangle$  is the sum of all basis Fock vectors in  $\mathcal{F}_0$ :*

$$|\Omega_0\rangle = \sum_{\lambda \in \mathcal{P}} |\lambda\rangle \quad (220)$$

where  $\lambda$  runs over the totality  $\mathcal{P}$  of all partitions  $\lambda = (\lambda_1, \lambda_2, \dots)$ , and  $|\lambda\rangle$  is defined as

$$|\lambda\rangle = (-)^{\beta_1 + \dots + \beta_k} \psi_{\alpha_1} \dots \psi_{\alpha_k} \psi_{-\beta_k-1}^\dagger \dots \psi_{-\beta_1-1}^\dagger |0\rangle \quad (221)$$

where we use the Frobenius notation for partitions [18]:  $\lambda = (\alpha_1, \dots, \alpha_k | \beta_1, \dots, \beta_k) = (\alpha | \beta)$  where  $\alpha_1 > \dots > \alpha_k \geq 0$ ,  $\beta_1 > \dots > \beta_k \geq 0$ ,  $k = 0, 1, 2, \dots$ <sup>6</sup>.

Then, we obtain the following simplest nontrivial DKP tau function (a version of a “vacuum tau function”):

**Proposition 14.** *We have the following DKP tau function*

$$\tau_o(\mathbf{t}) := \langle 0 | \Gamma(\mathbf{t}) | \Omega \rangle = \sum_{\lambda \in \mathcal{P}} s_\lambda(\mathbf{t}) = e^{\sum_{m=1}^{\infty} \frac{1}{2} m t_m^2 + \sum_{m=1}^{\infty} t_{2m-1}} \quad (222)$$

where  $s_\lambda(\mathbf{t})$  are Schur functions.

---

<sup>6</sup>Here and below  $k = 0$  will be related to  $\lambda = 0$ .

The second equality follows from the well-known formula [6]

$$\langle 0 | \Gamma(\mathbf{t}) | \lambda \rangle = s_\lambda(\mathbf{t}), \quad (223)$$

which is an example of the KP tau function. The third equality in (222) follows from the Exercise I-5-4 in [18] which should be re-written in terms of power sums.

**Corollary.** From consideration similar to [96] we obtain

$$e^{\frac{1}{2} \sum_{m=1}^{\infty} \frac{1}{m} J_{-m}^2 + \sum_{m=1,3,\dots}^{\infty} \frac{1}{m} J_{-m}} |l\rangle = \sum_{\lambda \in \mathbf{P}} |\lambda, l\rangle \quad (224)$$

Similar relations follows from each equation of Propositions 5 and 6.

Now, let us introduce

$$\tau_N(l, \mathbf{t}, U, \bar{A}, \bar{B}) := \langle N + l | \Gamma(\mathbf{t}) \mathbb{T}(U) g^+(\bar{B}) g^-(\bar{A}) | l \rangle \quad (225)$$

where  $g^-(\bar{A})$  and  $g^+(\bar{B})$  are given by (199),(200). Then by a direct evaluation of vacuum expectation value we obtain

$$S^{(2)}(\mathbf{t}; U, \bar{A}, \bar{B}) = \tau_0(0, \mathbf{t}, U, \bar{A}, \bar{B}) \quad (226)$$

which is the content of Proposition 2.

### 5.3 Mixed $lBKP$ and $sDKP$ tau function

.

$$\tau := \langle 0 | \Gamma(\mathbf{t}, \mathbf{t}', \mathbf{t}'') e^{\sum_{n,m \geq 0} (D_{nm} \phi_n \psi_m + D_{nm}^* \hat{\phi}_n \psi_{-m-1}^\dagger)} | 0 \rangle \quad (227)$$

and

$$\tau := \langle N | \Gamma(\mathbf{t}, \mathbf{t}') e^{\sum_{n,m \geq 0} D_{nm} \phi_n \psi_m} | 0 \rangle \quad (228)$$

$$= \sum_{h_1 > \dots > h_N \geq 0} \sum_{\substack{\alpha \in \mathbf{DP} \\ \ell(\alpha) \leq N}} 2^{-\frac{\ell(\lambda)}{2}} D_{\{\alpha, h\}} Q_\alpha(\frac{1}{2} \mathbf{t}') s_{\{h\}}(\mathbf{t}) \quad (229)$$

where

$$\Gamma(\mathbf{t}, \mathbf{t}', \mathbf{t}'') := \Gamma(\mathbf{t}) \Gamma_B(\mathbf{t}') \Gamma_B(\mathbf{t}''), \quad \Gamma(\mathbf{t}, \mathbf{t}') := \Gamma(\mathbf{t}) \Gamma_B(\mathbf{t}') \quad (230)$$

and where  $D_{\{\alpha, h\}}$  is given by (57).

The particular cases are ( $D_{nm} = e^{-U_n} \delta_{nm}$ )

$$\tau(\mathbf{t}, \mathbf{t}', U) = \sum_{\substack{\lambda \in \mathbf{P} \\ \ell(\alpha) \leq N}} 2^{-\frac{\ell(\lambda)}{2}} e^{-U_\lambda} s_\lambda(\mathbf{t}) Q_{\lambda^-}(\frac{1}{2} \mathbf{t}'), \quad (231)$$

$$\tau(\mathbf{t}, \mathbf{t}', U) = \sum_{\substack{\lambda \in \mathbf{P} \\ \ell(\alpha) \leq N}} 2^{-\frac{\ell(\lambda)}{2}} s_\lambda(\mathbf{t}) Q_{\lambda^-}(\frac{1}{2} \mathbf{t}'), \quad (232)$$

$$\tau(\mathbf{t}, U) = \sum_{\substack{\lambda \in \mathbf{P} \\ \ell(\alpha) \leq N}} e^{-U_\lambda} s_\lambda(\mathbf{t}) \quad (233)$$

$$\tau_0(\mathbf{t}) := \sum_{\substack{\lambda \in \mathbf{P} \\ \ell(\alpha) \leq N}} s_\lambda(\mathbf{t}) \quad (234)$$

where  $\lambda^-$  is the partition with shifted parts:  $\lambda_i^- := \lambda_i - i + N$ ,  $i = 1, \dots, N$ , and where

$$U_\lambda := \sum_{i=1}^N U_{\lambda_i - i} \quad (235)$$



## 5.4 Sums. Modifications of the Schur measure and tau functions. Discrete analogs of matrix ensembles

Here we plan to get use of the considered series in partitions

$$\sum_{\lambda \in \mathcal{P}} e^{-U_\lambda} s_\lambda(\mathbf{t}), \quad \sum_{\lambda \in \mathcal{P}} 2^{-\frac{\ell(\lambda)}{2}} e^{-U_\lambda} Q_{\lambda^-}(\tfrac{1}{2}\mathbf{t}'), \quad \sum_{\lambda \in \mathcal{P}} 2^{-\frac{\ell(\lambda)}{2}} e^{-U_\lambda} s_\lambda(\mathbf{t}) Q_{\lambda^-}(\tfrac{1}{2}\mathbf{t}'), \quad (236)$$

and also of the series

$$\sum_{\lambda \in \mathcal{P}} e^{-U_\lambda} s_\lambda(\mathbf{t}) s_\lambda(\bar{\mathbf{t}}), \quad \sum_{\lambda \in \mathcal{P}} 2^{-\ell(\lambda)} e^{-U_\lambda} Q_{\lambda^-}(\tfrac{1}{2}\mathbf{t}') Q_{\lambda^-}(\tfrac{1}{2}\bar{\mathbf{t}}') \quad (237)$$

earlier studied respectively in [30] and [31].

*Remark 19.* Let us notice that sums over partitions are studied in the context of random partitions. Representation theory Random partitions were started by the school of A. Vershik starting late 60-es and are under intensive studies nowadays (see series of papers by A. Vershik, S. Kerov, G. Olshansky, A. Okounkov, A. Borodin on this topic). The elegant fermionic approach to this subject was developed by A. Okounkov.

Let us write down most studied probability measures on the sets of partitions. By ideology developed by A. Vershik these are parts of representation theory of linear and symmetric groups.

The Plancherel measure on the set of partitions of  $n$  is defined as

$$n! \prod_{i=1}^N \frac{1}{(h_i!)^2} \prod_{1 \leq i < j \leq N} (h_i - h_j)^2, \quad n = |\lambda|, \quad h_i := \lambda_i - i + N$$

where  $N \geq \ell(\lambda)$ <sup>7</sup>. The  $z$ -measure is defined as

$$\frac{n!}{(zz')^n} \prod_{i=1}^N \frac{(z - i + 1)_{\lambda_i} (z' - i + 1)_{\lambda_i}}{(h_i!)^2} \prod_{1 \leq i < j \leq N} (h_i - h_j)^2, \quad n = |\lambda|, \quad h_i := \lambda_i - i + N$$

where  $z$  and  $z'$  are parameters, and  $(z)_k := \frac{\Gamma(z+k)}{\Gamma(z)}$  is the Pochhammer symbol.

The Schur measure on the set of all partitions was introduced by Okounkov in [73]. The weight of a partition  $\lambda$  is

$$W_\lambda(\mathbf{t}, \bar{\mathbf{t}}) = s_\lambda(\mathbf{t}) s_\lambda(\bar{\mathbf{t}})$$

where  $\mathbf{t}$  and  $\bar{\mathbf{t}}$  are parameters of the measure. The Schur measure generalizes the (poissonized with a parameter  $p$ ) Plancherel and  $z, z'$ -measures which may be basically obtained as evaluations of the Schur measure respectively at the points  $\mathbf{t} = \bar{\mathbf{t}} = e^{-p/2} \mathbf{t}_\infty$  and  $\mathbf{t} = e^{-p/2} \mathbf{t}(z), \bar{\mathbf{t}} = e^{-p/2} \mathbf{t}(z')$  in notations (380), (381). The poissonized Plancherel measure on the set of all partitions assign the weight

$$e^{-p|\lambda|} \prod_{i=1}^N \frac{1}{(h_i!)^2} \prod_{1 \leq i < j \leq N} (h_i - h_j)^2, \quad h_i := \lambda_i - i + N \quad (238)$$

The similarity of (238) to ensembles of random Hermitian matrices was observed and intensively worked out to solve combinatorial problems in late 90-es in papers by Okounkov, Borodin, Johansson and Baik, see [73], [78], [80] Earlier it was used in physics in [82], [83], [84]. Let us note that evaluation of the Schur measure at other points (382) and (383) yields links with different matrix models, see [32].

Now turn out to the studied series in partitions. They bring us to consider the following weights on the set of partitions

$$W_\lambda(\mathbf{t}, U) = e^{-U_\lambda} s_\lambda(\mathbf{t}), \quad W_\lambda(\mathbf{t}, \bar{\mathbf{t}}, U) = e^{-U_\lambda} s_\lambda(\mathbf{t}) s_\lambda(\bar{\mathbf{t}}), \quad (239)$$

if  $\lambda = (\alpha|\beta)$

$$W_\lambda(\mathbf{t}, \mathbf{t}', \bar{\mathbf{t}}, U) = 2^{-\ell(\lambda)} e^{-U_\lambda} s_\lambda(\mathbf{t}) Q_\alpha(\tfrac{1}{2}\mathbf{t}') Q_\beta(\tfrac{1}{2}\bar{\mathbf{t}}'), \quad (240)$$

and, in case partitions are restricted to have at most  $N$  parts:

$$W_\lambda(\mathbf{t}, \mathbf{t}', U) = 2^{-\frac{\ell(\lambda)}{2}} e^{-U_\lambda} s_\lambda(\mathbf{t}) Q_{\lambda^-}(\tfrac{1}{2}\mathbf{t}'), \quad (241)$$

---

<sup>7</sup>As one can check (238) does not depend on choice of  $N$  if  $N$  no less than the partition length  $\ell(\lambda)$ .

where  $\lambda^-$  is a strict partition defined by  $\lambda = (\lambda_1, \dots, \lambda_N)$  as  $\lambda^- := (\lambda_1 - 1 + N, \lambda_2 - 2 + N, \dots, \lambda_N)$ . On the set of strict partitions  $\alpha \in \text{DP}$

$$W_\alpha(\mathbf{t}', U) = 2^{-\frac{\ell(\alpha)}{2}} e^{-U_{\{\alpha\}}} Q_\alpha(\frac{1}{2}\mathbf{t}'), \quad W_\alpha(\mathbf{t}', \bar{\mathbf{t}}', U) = 2^{-\ell(\alpha)} e^{-U_{\{\alpha\}}} Q_\alpha(\frac{1}{2}\mathbf{t}') Q_\alpha(\frac{1}{2}\bar{\mathbf{t}}') \quad (242)$$

At last

$$W_\lambda(\mathbf{t}, U) = e^{-U_{\lambda \cup \lambda}} s_{\lambda \cup \lambda}(\mathbf{t}) \quad (243)$$

restricted on the set of partitions  $\lambda \cup \lambda = (\lambda_1, \lambda_1, \lambda_2, \lambda_2, \dots) \in \text{P}^2$  may be viewed as the weight of  $\lambda \in \text{P}$ .

Here  $e^{-U_\lambda}$  (or  $e^{-U_{\{\alpha\}}}$ ) plays the role of additional Gibbs-Boltzmann weight assigned to each configuration  $\lambda$  (or  $\alpha$ ) induced by external sources. Let us note that in  $U$ -dependence of partition functions  $Z = Z(U)$  shows non-analytic behavior (phase transitions, compare to [82], [83], [86], [69], or just to, say,  ${}_1F_0(a) = (1-x)^{-a}$  behavior which is the simplest example of the series under consideration). Now,  $W_\lambda(\mathbf{t}, \bar{\mathbf{t}}, 0)$  is the Schur measure studied by Okounkov in [73] while  $W_\alpha(\mathbf{t}', \bar{\mathbf{t}}', 0)$  is the shifted Schur measure introduced and studied by Tracy and Widom in [87].

Then all normalization functions (in physics: "partition functions")  $Z = \sum_\lambda W_\lambda$  of these ensembles of random partitions are tau functions (236) and (237). Provided the weights are non-negative the probability of a configuration  $\lambda$  is

$$p_\lambda = \frac{W_\lambda}{Z}$$

where for each ensemble  $Z$  is a tau function.

It is interesting, that if we put all  $\mathbf{t}, \bar{\mathbf{t}}, \mathbf{t}', \bar{\mathbf{t}}'$  to be equal to  $(1, 0, 0, \dots)$  and deform  $U$  via deformation parameters  $\mathbf{t}^*$  according to (244) below we obtain that the partition function  $Z = Z(\mathbf{t}^*)$  is again a tau function, in this case, it is a tau function with respect to the deformation parameters  $\mathbf{t}^*$ . Moreover, it will be related to discrete versions of ensembles of random matrices (where in context of considerations in physics parameters  $\mathbf{t}^*$  are commonly called coupling constants).

The last remark is the following. It is natural to consider bi-measure on pairs of partitions taking general tau functions as weight functions. Such models should possess good properties. Tau functions of 2-KP, 2-IBKP ((2)), 2-sBKP ((400)) and IBKP coupled to sDKP provide respectively  $\text{P} \times \text{P}$ ,  $\text{DP} \times \text{DP}$ ,  $\text{DP} \times \text{P}$  with bi-measures according to formulae (3),(2), (398), (229). Objects of these models like correlation functions should be expressed in terms of Baker-Akhiezer functions.

Now we shall write down normalization functions  $Z$  for ensembles of random partitions with measures  $W_\lambda$  according to (239)-(242). Parameters  $\mathbf{t}, \bar{\mathbf{t}}, \mathbf{t}', \bar{\mathbf{t}}'$  will be equated to one of (380), (381),(382),(383) yielding different specifications of measures (239)-(242).

**(i) Ensemble with the measure  $W_\lambda(\mathbf{t}, U(\mathbf{t}^*))$ . Discrete  $\beta = 1$  ensembles.** The partition function  $Z$  of this ensemble is the tau function (214).

First we put  $\mathbf{t} = e^p \mathbf{t}_\infty$  taking

$$\mathbf{t}_\infty = (1, 0, 0, \dots), \quad U_n = U_n(\mathbf{t}^*) = U_n^{(0)} - \sum_{m=1}^{\infty} n^m t_m^* \quad (244)$$

Then due to (384) the weight function is

$$W_\lambda^{(1)}(\mathbf{t}^*) = \prod_{n < m \leq N} (h_n - h_m) \prod_{j=1}^N \frac{1}{h_j!} e^{\sum_{n=1}^{\infty} t_n^* h_j^n + p h_j - U_{h_j}^{(0)}} \quad (245)$$

the normalization function  $Z = Z^{(1)}$  is equal to

$$Z^{(1)}(\mathbf{t}^*, p) = \tau_N(\mathbf{t}_\infty, U(\mathbf{t}^*)) = \frac{e^{c_N p}}{N!} \sum_{h_1, \dots, h_N=0}^{\infty} \prod_{n < m \leq N} |h_n - h_m| \prod_{j=1}^N e^{\sum_{n=1}^{\infty} t_n^* h_j^n} \frac{e^{p h_j - U_{h_j}^{(0)}}}{h_j!} \quad (246)$$

where the factor  $N!$  appears when we spread the summation over the cone  $h_1 > \dots > h_N \geq 0$  to the independent summation over each of  $h_j = 0, 1, 2, \dots$  at the same time changing  $\prod_{n < m \leq N} (h_n - h_m)$  to  $\prod_{n < m \leq N} |h_n - h_m|$ .  $c_N$  is an unrelated constant. The parameter of Poissonization  $p$  can be identified with  $t_1^*$ .

If we take the same  $U$  as in (244) but take  $\mathbf{t}$  as in (381) we obtain an analog of the Poissonized  $z$ -measure (with  $z = a$ ):

$$W_\lambda^{(2)}(\mathbf{t}^*, z) = c_N(a) \prod_{n < m \leq N} (h_n - h_m) \prod_{j=1}^N \frac{(a)_{h_j+1-N}}{h_j!} e^{\sum_{n=1}^\infty t_n^* h_j^n - U_{h_j}^{(0)}} \quad (247)$$

with the normalization function

$$Z^{(2)}(\mathbf{t}^*) = \tau_N(\mathbf{t}(a), U(\mathbf{t}^*)) = \frac{c_N(a)}{N!} \sum_{h_1, \dots, h_N=0}^\infty \prod_{n < m \leq N} |h_n - h_m| \prod_{j=1}^N e^{\sum_{n=1}^\infty t_n^* h_j^n} \frac{e^{-U_{h_j}^{(0)}}}{h_j!} \quad (248)$$

where  $c_N(a)$  is given by (323).

*Remark 20.* Both expressions (246) and (246) which are BKP tau functions evaluated at points  $\mathbf{t} = \mathbf{t}_\infty$  and  $\mathbf{t} = \mathbf{t}(a)$  are also BKP tau functions with respect to new parameters  $\mathbf{t}^*$  introduced in (244). See also Section 6 about interlinks between different BKP tau functions. At the same time these tau functions where  $\mathbf{t}^*$  are higher times are examples of the  $\beta = 1$  ensemble [17] where the measure is proportional to a sum of delta functions and may be treated as a discrete version of ensemble of random orthogonal matrices with positive eigenvalues, where the measure deformation parameters are  $\mathbf{t}^*$ .

Next, instead of (244) we put

$$t_m = \frac{1}{m} \frac{1 - (q^a)^m}{1 - q^m}, \quad U_n = U_n(\mathbf{t}^*; q) = U_n^{(0)} - \sum_{m=-\infty}^\infty q^{nm} t_m^* \quad (249)$$

By (383) we obtain a different model with the weight

$$W_\lambda^{(3)}(\mathbf{t}^*; q, a) = c_N(a, q) \prod_{n < m \leq N} (q^{h_n} - q^{h_m}) \prod_{j=1}^N \frac{(q^a; q)_{h_i-n+1}}{(q; q)_{h_j}} e^{\sum_{n=-\infty}^\infty t_n^* q^{nh_j} - U_{h_j}^{(0)}} \quad (250)$$

with the following normalization function

$$\begin{aligned} Z^{(3)}(\mathbf{t}^*; a, q) &= \tau_N(\mathbf{t}(a, q), U(\mathbf{t}^*)) = \\ &= \frac{c_N(a, q)}{N!} \sum_{h_1, \dots, h_N=0}^\infty \prod_{n < m \leq N} |q^{h_n} - q^{h_m}| \prod_{j=1}^N \frac{(q^a; q)_{h_i-n+1}}{(q; q)_{h_j}} e^{\sum_{n=-\infty}^\infty t_n^* q^{nh_j} - U_{h_j}^{(0)}} \end{aligned} \quad (251)$$

which, for  $q \in S^1$ , may be considered as a discrete analogue of the circulate  $\beta = 1$  ensemble ([17]).  $c_N(a, q)$  is defined in (323). Let us choose the limit  $q^a \rightarrow 0$  (or, the same,  $\mathbf{t}$  is chosen by (382)) then  $(q^a; q)_{h_i-n+1} \rightarrow 1$ .

*Remark 21.* Expression (251) is the BKP tau function evaluated at point  $\mathbf{t} = \mathbf{t}(a, q)$  is also a 2-BKP tau function (197) with respect to new parameters  $\{t_n^*, n > 0\}$  and  $\{t_n^*, n < 0\}$  introduced in (249) and at the same time for  $q \in S^1$  is an example of the  $\beta = 1$  circular ensemble [17] where the measure is singular and is equal to a weighted sum of delta functions where the weight parameters depends on  $\mathbf{t}^*$ .

**(ii) Ensemble with the measure  $W_\lambda(\mathbf{t}, \bar{\mathbf{t}}, U(\mathbf{t}^*))$  of (239). Discrete  $\beta = 2$  ensembles.** The partition function  $Z$  of this ensemble is the KP tau function (203). This case was considered in the paper [32], here we only write down the most general case where  $\mathbf{t} = \mathbf{t}(a, q)$ ,  $\bar{\mathbf{t}} = \mathbf{t}(a', q)$  (which may be treated as a  $q$ -version of  $z, z'$ -measure, with  $z = a, z' = a'$ ) and where  $U$  are chosen by (249):

$$\begin{aligned} W_\lambda^{(4)}(\mathbf{t}^*; a, a', q) &= \\ c_N(a, q) c_N(a', q) \prod_{n < m \leq N} (q^{h_n} - q^{h_m})^2 \prod_{j=1}^N \frac{(q^a; q)_{h_i-n+1} (q^{a'}; q)_{h_i-n+1}}{((q; q)_{h_j})^2} e^{\sum_{n=-\infty}^\infty t_n^* q^{nh_j} - U_{h_j}^{(0)}} \end{aligned} \quad (252)$$

with the following normalization function

$$Z^{(4)}(\mathbf{t}^*; a, a', q) =$$

$$\frac{c_N(a, q)c_N(a', q)}{N!} \sum_{h_1, \dots, h_N=0}^{\infty} \prod_{n < m \leq N} (q^{h_n} - q^{h_m})^2 \prod_{j=1}^N \frac{(q^a; q)_{h_i-n+1} (q^{a'}; q)_{h_i-n+1}}{((q; q)_{h_j})^2} e^{\sum_{n=-\infty}^{\infty} t_n^* q^{nh_j} - U_{h_j}^{(0)}} \quad (253)$$

The last expression which is the 2-KP (TL) tau function (203) at the same time is 2-KP tau function with respect to the variables  $\{t_n^*, n > 0\}$  and  $\{t_n^*, n < 0\}$ . For  $q \in S^1$  it may be identified with a discrete version of the one-matrix model of unitary matrices [41].

**(iii) Ensemble (240):**  $W_\lambda(\mathbf{t}, \mathbf{t}', \bar{\mathbf{t}}', U(\mathbf{t}^*))$  where  $\mathbf{t} = \mathbf{t}' = \bar{\mathbf{t}}' = \mathbf{t}_\infty$ .

This ensemble is defined on the set of all partitions  $\lambda = (\alpha|\beta) \in \mathcal{P}$ ,

$$W_\lambda^{(5)}(\mathbf{t}^*) = \prod_{i>j}^k \frac{(\alpha_i - \alpha_j)^2 (\beta_i - \beta_j)^2}{(\alpha_i + \alpha_j)(\beta_i + \beta_j)} \prod_{i,j=1}^k \frac{1}{\alpha_i + \beta_j + 1} \prod_{i=1}^k \frac{e^{U_{-\beta_{i-1}}(\mathbf{t}^*) - U_{\alpha_i}(\mathbf{t}^*)}}{(\alpha_i! \beta_i!)^2} \quad (254)$$

$$Z^{(5)}(\mathbf{t}^*) = 1 + \sum_{k=1}^{\infty} \sum_{\alpha} \sum_{\beta} \prod_{i=1}^k \frac{e^{U_{\{\beta\}}(\mathbf{t}^*) - U_{\{\alpha\}}(\mathbf{t}^*)}}{(\alpha_i! \beta_i!)^2} \prod_{i>j}^k \frac{(\alpha_i - \alpha_j)^2 (\beta_i - \beta_j)^2}{(\alpha_i + \alpha_j)(\beta_i + \beta_j)} \prod_{i,j=1}^k \frac{1}{\alpha_i + \beta_j + 1} \quad (255)$$

where summation ranges over all pairs of strict partitions  $\alpha$  and  $\beta$  such that  $\ell(\alpha) = \ell(\beta)$ .

One can convert the summations over the cones  $\alpha_1 > \dots > \alpha_k \geq 0$ ,  $\beta_1 > \dots > \beta_k \geq 0$  to the summation over independent numbers  $\alpha_i$  and  $\beta_i$  as follows

$$Z^{(5)}(\mathbf{t}^*) = 1 + \sum_{k=1}^{\infty} \frac{1}{k!} \sum_{\alpha_1, \dots, \alpha_k=0}^{\infty} \sum_{\beta_1, \dots, \beta_k=0}^{\infty} \prod_{i=1}^k \frac{e^{U_{\{\beta\}}(\mathbf{t}^*) - U_{\{\alpha\}}(\mathbf{t}^*)}}{(\alpha_i! \beta_i!)^2} \prod_{i>j}^k \frac{(\alpha_i - \alpha_j)(\beta_i - \beta_j)}{(\alpha_i + \alpha_j)(\beta_i + \beta_j)} \prod_{i=1}^k \frac{1}{\alpha_i + \beta_i + 1} \quad (256)$$

**(iv) Ensemble (241):**  $W_\lambda(\mathbf{t}, \mathbf{t}', U(\mathbf{t}^*))$  where for  $U$  and  $\mathbf{t} = \mathbf{t}' = \mathbf{t}_\infty$  see (244). It is defined on partitions  $\lambda$  whose length do not exceed a given number  $N$ . Below  $h_i = \lambda_i - i + N$ .

$$W_\lambda^{(6)}(\mathbf{t}^*) = \prod_{1 \leq i < j \leq N} \frac{(h_i - h_j)^2}{h_i + h_j} \prod_{i=1}^N \left( \frac{1}{h_i!} \right)^2 e^{-U_{h_i}(\mathbf{t}^*)} \quad (257)$$

$$\begin{aligned} Z^{(6)}(\mathbf{t}^*) &= \tau_N(\mathbf{t}, U, A) = \sum_{h_1 > \dots > h_N} \prod_{i < j} \frac{(h_i - h_j)^2}{h_i + h_j} \prod_{i=1}^N \left( \frac{1}{h_i!} \right)^2 e^{-U_{h_i}} \\ &= \frac{1}{N!} \sum_{h_1, \dots, h_N=0}^{\infty} \prod_{i < j} \frac{(h_i - h_j)^2}{h_i + h_j} \prod_{i=1}^N \left( \frac{1}{h_i!} \right)^2 e^{-U_{h_i}} \end{aligned} \quad (258)$$

Actually this choice of  $A$  in (206) is exactly related to the Example (212) which is the tau function of mixed IBKP-sDKP tau function. Indeed if  $\mathbf{t}' = \mathbf{t}_\infty := (1, 0, 0, \dots)$  then it is known that

$$Q_\lambda(\frac{1}{2}\mathbf{t}_\infty) = 2^{\frac{\ell(\lambda)}{2}} \prod_{i=1}^k \frac{1}{\lambda_i!} \prod_{i < j}^k \frac{\lambda_i - \lambda_j}{\lambda_i + \lambda_j}$$

see (393) in the Appendix.

In the spirit of discrete-continuous duality one may expected that there is a continuous counterpart to (258). Indeed BKP-sDKP tau function may be chosen as follows

$$\tau_N^{IBKP-sDKP}(\mathbf{t}, \mathbf{t}', U) = \langle N + l, 0 | \Gamma(\mathbf{t}) \Gamma'(\mathbf{t}') e^{\int \psi(x) \phi(x) d\mu(x)} \Gamma(\mathbf{t}) \Gamma'(\mathbf{t}') | l, 0 \rangle \quad (259)$$

$$= \int \dots \int \frac{(x_i - x_j)^2}{x_i + x_j} \prod_{i=1}^N d\mu(x_i, l, \mathbf{t}, \bar{\mathbf{t}}, \mathbf{t}', \bar{\mathbf{t}}') \quad (260)$$

where  $l$ , is the Dirac sea level of IBKP vacuum vector,  $\mathbf{t}$  and  $\bar{\mathbf{t}}$  are higher times of the coupled IBKP, and  $\mathbf{t}', \bar{\mathbf{t}}'$  are higher times for coupled sBKP. These "times" play the role of deformation parameters:

$$d\mu(x, l, \mathbf{t}, \bar{\mathbf{t}}, \mathbf{t}', \bar{\mathbf{t}}') = \frac{x^l}{\sqrt{2}} \exp \sum_{n=1}^{\infty} (x^n t_n + x^{2n-1} t'_{2n-1} - x^{-n} \bar{t}_n - x^{1-2n} \bar{t}'_{2n-1})$$

Similar expressions appear in the study of random density matrices and Bures densities [36].

**(v) Ensembles (242):**  $W_\lambda(\mathbf{t}, \mathbf{t}', \bar{\mathbf{t}}, U(\mathbf{t}^*))$  where  $\mathbf{t} = \mathbf{t}' = \bar{\mathbf{t}}' = \mathbf{t}_\infty$ . These ensembles are defined on the set of all strict partitions  $\alpha = (\alpha_1, \alpha_2, \dots) \in \text{DP}$ .

Ensembles related to (242) and their continues versions were considered in [26] as examples of sBKP tau functions. Ensembles  $W_\alpha(\mathbf{t}', U)$  where  $\mathbf{t}' = \mathbf{t}_\infty$  and  $W_\alpha(\mathbf{t}', \bar{\mathbf{t}}, U)$  with  $\mathbf{t}' = \bar{\mathbf{t}}' = \mathbf{t}_\infty$  are written respectively as

$$W^{(7)}(\mathbf{t}^*) = \prod_{i < j} \frac{\alpha_i - \alpha_j}{\alpha_i + \alpha_j} \prod_{i=1}^n \frac{1}{\alpha_i!} e^{-U_{\{\alpha_i\}}(\mathbf{t}^*)} \quad (261)$$

$$Z^{(7)}(\mathbf{t}^*) = 1 + \sum_{n=1}^{\infty} \frac{1}{n!} \sum_{\alpha_1, \dots, \alpha_n=0}^{\infty} \prod_{i < j} \frac{\alpha_i - \alpha_j}{\alpha_i + \alpha_j} \prod_{i=1}^n \frac{1}{\alpha_i!} e^{-U_{\{\alpha_i\}}(\mathbf{t}^*)} \quad (262)$$

where

$$U_{\{\alpha_i\}}(\mathbf{t}^*) = \sum_{i=1}^{\ell(\alpha)} U_{\alpha_i}(\mathbf{t}^*), \quad U_n(\mathbf{t}^*) := U_n^{(0)} + \sum_{i=1,3,\dots} n^i t_i^*, \quad U_n^{(0)} = U_{-n}^{(0)} \quad (263)$$

and ensembles related to the shifted Schur measure introduced in [87]

$$W^{(8)}(\mathbf{t}^*) = \prod_{i < j} \left( \frac{\alpha_i - \alpha_j}{\alpha_i + \alpha_j} \right)^2 \prod_{i=1}^n \left( \frac{1}{\alpha_i!} \right)^2 e^{-U_{\{\alpha_i\}}(\mathbf{t}^*)} \quad (264)$$

$$Z^{(8)}(\mathbf{t}^*) = 1 + \sum_{n=1}^{\infty} \frac{1}{n!} \sum_{\alpha_1, \dots, \alpha_n=0}^{\infty} \prod_{i < j} \left( \frac{\alpha_i - \alpha_j}{\alpha_i + \alpha_j} \right)^2 \prod_{i=1}^n \left( \frac{1}{\alpha_i!} \right)^2 e^{-U_{\{\alpha_i\}}(\mathbf{t}^*)} \quad (265)$$

The last expression may be identified with a KdV tau function where  $\mathbf{t}^*$  are KdV higher times (compare with [85]). Formula (262) is a sBKP tau function with respect to variables  $\mathbf{t}^*$  introduced by (263).

**(vi) Ensemble (239):**  $W_\lambda(\mathbf{t}, \bar{\mathbf{t}}, U(\mathbf{t}^*))$  where for  $U$  and for  $\mathbf{t} = \mathbf{t}_\infty$  see (244), and where

$$\bar{t}_m(x) = \frac{1}{m} \sum_{i=1}^N x_i^m$$

The weight

$$W^{(9)}(\mathbf{t}^*, x) = W_\lambda(\mathbf{t}_\infty, \bar{\mathbf{t}}(x), U(\mathbf{t}^*)) \quad (266)$$

is defined on partitions  $\lambda$  whose length do not exceed a given number  $N$ . Below  $h_i = \lambda_i - i + N$ .

Then the normalization function is

$$Z^{(9)}(\mathbf{t}^*, x) = \sum_{h_1=0}^{\infty} \cdots \sum_{h_N=0}^{\infty} (h_i - h_j) \prod_{i=1}^N \frac{1}{h_i!} x_i^{h_i} e^{-U_{h_i}} \quad (267)$$

which may be considered as a discrete version of the so-called one-matrix model with a source:

$$\int e^{\sum_{m=1}^{\infty} \text{Tr } H^m t_m^* + \text{Tr } \Lambda H} dH = \int_{\mathbb{R}^N} \Delta_N(z) \prod_{i=1}^N e^{\sum z_i^m t_m^* + \sum_{i=1}^N \lambda_i z_i} dz_i$$

where  $z_i$  and  $\lambda_i$  are respectively eigenvalues of the Hermitian matrix  $H$  and the normal matrix  $\Lambda$ .

**(vii) Ensemble (239):**  $W_\lambda(\mathbf{t}, \bar{\mathbf{t}}, U(\mathbf{t}^*))$  where for  $U$  and for  $\mathbf{t} = \mathbf{t}(\infty, q)$ , see (249), and where

$$\bar{t}_m(x) = \frac{1}{m} \sum_{i=1}^N x_i^m$$

The weight

$$W^{(10)}(\mathbf{t}^*, x) = W_\lambda(\mathbf{t}(q), \bar{\mathbf{t}}(x), U(\mathbf{t}^*)) \quad (268)$$

is defined on partitions  $\lambda$  whose length do not exceed a given number  $N$ . Below  $h_i = \lambda_i - i + N$ .

Then the normalization function is

$$Z^{(10)}(\mathbf{t}^*, x) = \sum_{h_1=0}^{\infty} \cdots \sum_{h_N=0}^{\infty} (q^{h_i} - q^{h_j}) \prod_{i=1}^N \frac{1}{(q; q)_{h_i}} x_i^{h_i} e^{-U_{h_i}(\mathbf{t}^*)} \quad (269)$$

which in case  $q \in S^1$  may be considered as a discrete version of models of a random unitary matrices with a source:

$$\oint e^{\sum_{m=-\infty}^{\infty} \text{Tr } U^m t_m^* + \text{Tr } \Lambda U} dU = \oint \Delta_N(z) \prod_{i=1}^N e^{\sum_{m \in \mathbb{Z}} e^{m z_i} t_m^* + \sum_{i=1}^N \lambda_i z_i} \frac{dz_i}{z_i}$$

where  $e^{z_i}$  and  $\lambda_i$  are respectively eigenvalues of the unitary matrix  $U$  and the source  $\Lambda$ .

**(viii) Ensemble (241):**  $W_\lambda(\mathbf{t}, \mathbf{t}', U(\mathbf{t}^*))$  where  $\mathbf{t}' = \mathbf{t}_\infty$  and where

$$t_m(x) = \frac{1}{m} \sum_{i=1}^N x_i^m$$

The weight

$$W^{(11)}(\mathbf{t}^*, x) = W_\lambda(\mathbf{t}(x), \mathbf{t}_\infty, U(\mathbf{t}^*)) \quad (270)$$

is defined on partitions  $\lambda$  whose length do not exceed a given number  $N$ . Below  $h_i = \lambda_i - i + N$ .

The normalization function is

$$Z^{(11)}(\mathbf{t}^*, x) = \sum_{h_1, \dots, h_N=0}^{\infty} \frac{h_i - h_j}{h_i + h_j} \prod_{i=1}^N \frac{1}{h_i!} x_i^{h_i} e^{-U_{h_i}} \quad (271)$$

which is a discrete version version of the following 2-sBKP tau function [26].

$$Z^{(10)} = \int_{\mathbb{R}^N} \frac{z_i - z_j}{z_i + z_j} \prod_{i=1}^N e^{\sum_{m=1,3,\dots}^{\infty} (t_m^* z_i^m + \bar{t}_m^* z_i^{-m}) + \lambda_i z_i} d\mu(z_i)$$

**(ix) Ensembles (243). Discrete  $\beta = 4$  ensemble:** Now take  $W_\lambda(\mathbf{t}, U(\mathbf{t}^*))$  of (243) where we recall  $\lambda = (\lambda_1, \lambda_1, \dots, \lambda_n, \lambda_n)$  and  $\mathbf{t}$  is either  $\mathbf{t}_\infty$  or  $\mathbf{t}(q)$ . In the first case,  $\mathbf{t} = \mathbf{t}_\infty$ , we have

$$W_\lambda^{(12)}(\mathbf{t}^*) = \tilde{\Delta}_n(x)^4 \prod_{i=1}^n \frac{e^{-U_{x_i}(\mathbf{t}^*) - U_{x_{i-1}}(\mathbf{t}^*)}}{x_i!(x_i - 1)!} \quad (272)$$

where the set

$$x_i := \lambda_i - 2i + 2n, \quad i = 1, \dots, n \quad (273)$$

$$\tilde{\Delta}_n(x)^4 := \prod_{1 \leq i < j \leq n} (x_i - x_j)^2 ((x_i - x_j)^2 - 1) \quad (274)$$

The grand partition function is a discrete version of grand partition function of the symplectic ensemble

$$Z^{(12)} = \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{x_1, \dots, x_n} \tilde{\Delta}_n(x)^4 \prod_{i=1}^n \frac{e^{-U_{x_i}(\mathbf{t}^*) - U_{x_{i-1}}(\mathbf{t}^*)}}{x_i!(x_i - 1)!} \quad (275)$$

In the second case,  $\mathbf{t} = \mathbf{t}(q)$ , we have

$$W_\lambda^{(12)}(\mathbf{t}^*) = \tilde{\Delta}_n(q^x)^4 \prod_{i=1}^n \frac{e^{-U_{x_i}(\mathbf{t}^*) - U_{x_{i-1}}(\mathbf{t}^*)}}{(q; q)_{x_i} (q; q)_{x_i - 1}} \quad (276)$$

where the set of  $\{x_i\}$  is the same as before while

$$\tilde{\Delta}_n(q^x)^4 := q^{-1} \prod_{1 \leq i < j \leq n} (q^{x_i} - q^{x_j})^2 (q^{x_i - 1} - q^{x_j})(q^{x_i} - q^{x_j - 1}) \quad (277)$$

The grand partition function is the grand partition function for a the following  $\beta = 4$  ensemble

$$Z^{(12)} = \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{x_1, \dots, x_n} \tilde{\Delta}_n(q^x)^4 \prod_{i=1}^n \frac{e^{-U_{x_i}(\mathbf{t}^*) - U_{x_{i-1}}(\mathbf{t}^*)}}{(q; q)_{x_i} (q; q)_{x_i - 1}} \quad (278)$$

## 6 Interlinks between tau functions of IBKP

Here we will show that for  $\mathbf{t}^{(i)}$  be one of (99)-(102) (see also Appendix E.4) . and for  $U = U(\mathbf{t}^*)$  be specified later there exist relations between different IBKP tau functions as follows:

$$\langle l' | \Gamma(\mathbf{t}^{(i)}) \mathbb{T}(\mathbf{t}^*) g | l \rangle = \langle l' - l | \Gamma(\mathbf{t}_+^*) g_i^* \Gamma(\mathbf{t}_-^*) | 0 \rangle \quad (279)$$

where  $g_i^*$  is constructed in terms of a given  $g$  according to a choice of  $\mathbf{t}^{(i)}$ , and where  $\mathbf{t}_+^*$  is the collection  $\{t_m^*, m > 0\}$  while  $\mathbf{t}_-^*$  is the collection  $\{t_m^*, m < 0\}$ .

Here, in case  $\mathbf{t}^{(i)}$  is chosen either via (380) or via (381), the higher times  $\mathbf{t}^*$  are related to variables  $U$  by (105), namely

$$U_n = U_n(\mathbf{t}^*, \bar{\mathbf{t}}^*) = U_n^{(0)} + \sum_{m=1}^{\infty} \left( \frac{an+b}{cn+d} \right)^m t_m^* + t_0^* \ln \left( \frac{an+b}{cn+d} \right) - \sum_{m=1}^{\infty} \left( \frac{an+b}{cn+d} \right)^{-m} \bar{t}_m^*$$

where  $a, b, c, d$  are parameters chosen arbitrary in a way we have no singular terms in the above sum. In particular, if we do not want to have dependence on  $\mathbf{t}_-^*$  parameters the  $U$ -dependence may be chosen as

$$U_n = U_n(\mathbf{t}^*, c) = U_n^{(0)} - \sum_{m>0} n^m t_m^* \quad (280)$$

as it was done in, say, [56], or in [47], for the different case, for a TL tau function. while

In case (101), or (102) by

$$U_n = U_n(\mathbf{t}^*; q, a, b, c, d) = U_n^{(0)} - \sum_{m>0} \left( \frac{aq^n + b}{cq^n + d} \right)^m t_m^* - \sum_{m<0} \left( \frac{aq^n + b}{cq^n + d} \right)^m t_m^* - t_0^* \ln \left( \frac{aq^n + b}{cq^n + d} \right) \quad (281)$$

In particular case we may take

$$U_n = U_n(\mathbf{t}^*; q) = U_n^{(0)} - \sum_{m>0} q^{nm} t_m^* \quad (282)$$

as in [52] or [28].

Now we equate a IBKP tau functions depending on parameters  $U$  and thus depending on the parameters  $\mathbf{t}^*$  to a certain multisoliton 2-IBKP tau functions where  $\mathbf{t}^*$  play the role of higher times. Thus we present a sort of duality between the  $U$  and  $\mathbf{t}$  variables<sup>8</sup>. This link may be compared with [28] (see also references there).

Actually this section explains "dualities" between discrete and continuous expressions for tau functions, see subsection 5.4. Such dualities are also considered in [22].

Let us choose (244). Now, in Frobenius notation  $\lambda = (p_1, \dots, p_k | -q_1 - 1, \dots, -q_k - 1)$ , we have (388):

$$s_\lambda(p, 0, 0, \dots) = \prod_{n < m \leq k} \frac{(p_n - p_m)(q_n - q_m)}{(p_n - q_m)(p_m - q_n)} \left( \prod_{n=1}^k \frac{(p_n - q_n)^{-1}}{\Gamma(p_n + 1)\Gamma(-q_n)} \right) \quad (283)$$

(where numbers  $p_n$  and  $q_n$  are related to the Frobenius coordinates as  $p_n = \alpha_n$  and  $q_n = -\beta_n - 1$ <sup>9</sup>). Written in this form the Schur function may be interpreted as a familiar formula for shift of solitons due to interaction where factor in bracket is irrelevant and may be included to the choice of initial position of solitons. Therefore

$$\begin{aligned} \tau(\mathbf{t}, U(\mathbf{t}^*)) &= \sum_{\lambda \in \mathbf{P}} e^{-U_\lambda(\mathbf{t}^*)} s_\lambda(\mathbf{t}) = \\ &= \sum_{k=0} \sum_{\{p,q\}_k} \prod_{n < m \leq k} \frac{(p_n - p_m)(q_n - q_m)}{(p_n - q_m)(p_m - q_n)} \left( \prod_{n=1}^k \frac{(p_n - q_n)^{-1}}{\Gamma(p_n + 1)\Gamma(-q_n)} \right) e^{\sum_{n=1}^{\infty} t_n^* (p_n^n - q_n^n)} \end{aligned} \quad (284)$$

<sup>8</sup>Here we convert an observation of [27] to the case of IBKP hierarchy.

<sup>9</sup>or, more generally one can take  $p_n = \frac{a\alpha_n + b}{c\alpha_n + d}$ ,  $q_n = \frac{a(-\beta_n - 1) + b}{c(-\beta_n - 1) + d}$  with arbitrary  $a, b, c, d$  which keeps linear fraction factor in (283) [27].

where  $\{p, q\}_k$  means summation over sets of integers  $p_1 > \dots > p_k \geq 0 > q_k > \dots > q_1$  which may be interpreted as a IDKP multisoliton tau function,  $\tau^*(\mathbf{t}^*)$ , where higher times,  $\mathbf{t}^*$ , are related to  $U = U(\mathbf{t}^*)$  of original IBKP tau function. The fermionic expression for this multisoliton tau function is

$$\tau^*(\mathbf{t}^*) := \langle 0 | e^{J(\mathbf{t}^*)} g^-(\bar{A}, \mathbf{p}) g^+(\bar{B}, \mathbf{q}) | 0 \rangle = \tau(\mathbf{t}, U) \quad (285)$$

where operators  $g^-(\bar{A}, \mathbf{p}), g^+(\bar{B}, \mathbf{q})$  coincide respectively with  $g^-(\bar{A}), g^+(\bar{B})$  if in the last group we replace each fermionic Fourier mode  $\psi_n, \psi_n^\dagger$  respectively by  $\psi(p_n), \psi^\dagger(q_n)$ :

## 7 Partition functions for certain random processes

A number of random processes may be described via a sort of partition function  $Z$  as follows. We have an un-normalized weight  $W_{\mu \rightarrow \nu}(\mathbf{T})$  for a transition from a state  $\mu$  to a state  $\lambda$  during a time interval  $\mathbf{T}$ . To get the probability for this transition,  $p_{\mu \rightarrow \nu}(\mathbf{T})$ , we divide this weight by the normalization function  $Z_\mu(\mathbf{T})$  which is equal to the sum of weights to get any state during elapse of time starting from the state  $\mu$ , then,

$$p_{\mu \rightarrow \lambda}(\mathbf{T}) = \frac{1}{Z_\mu(\mathbf{T})} W_{\mu \rightarrow \lambda}(\mathbf{T}), \quad Z_\mu(\mathbf{T}) = \sum_{\lambda} W_{\mu \rightarrow \lambda}(\mathbf{T})$$

where sum runs over all possible states  $\lambda$  which may be achieved from initial state  $\mu$  during a time interval  $\mathbf{T}$ . The normalization function  $Z$  provides the condition that the sum of probabilities is equal to unity.

In case the state of a system may be identified with basis Fock vectors  $\langle \mu |$  and the transition weight  $W_{\mu \rightarrow \lambda}(\mathbf{T})$  may be written as a matrix element of some given operator  $o(\mathbf{T})$  which acts in Fock space:

$$W_{\mu \rightarrow \lambda}(\mathbf{T}) := \langle \mu | o(\mathbf{T}) | \lambda \rangle$$

then, to get the partition function  $Z_\mu(\mathbf{T})$  we need to evaluate

$$Z_\mu(\mathbf{T}) = \langle \mu | o(\mathbf{T}) | \Omega_0 \rangle$$

Examples include random turn vicious walkers model [70] (model (B) in section 4) which may be treated as a modification of ‘exclusion processes. Versions of this model were considered in [81], [76], [80]. Relations to

[88] (section 4.1), see also [89]. In these version walkers move in one direction, then

Recently, we found a mentioning of the fermionic approach to this problem in the paper by Okounkov [50] (Section 1.4.4.). Below we present an explicit evaluation of the probability to get a given final position of the walkers in the simplest version of this model where we shall use IDKP tau function (222).

**Example 1.** We are interested in creating a Young diagram (YD)  $\lambda$  by gluing box by box in a way that each intermediate figure is a Young diagram. It means that at each step we can glue a box only to a certain number of admissible places on the boundary of a diagram. A consequence of Young diagrams may be called the path connecting initial and final Young diagrams. The number of YD along a path will be called the length of the path. For simplicity we take an empty YD (YD without nodes) as the initial one. Now one can address few questions: (1) what is the number of paths of length  $\mathbf{T}$  starting at empty YD and ending at a given YD  $\lambda$ , (this number will be denoted by  $W_{0 \rightarrow \lambda}(\mathbf{T})$ ). (2) what is the number of paths starting at empty YD (empty state relates to  $\mu = 0$ ) of a length  $\mathbf{T}$  (this number will be denoted by  $Z_0(\mathbf{T})$ ). If we consider this creation of a YD as a random process describing the gluing of the boxes, then the probability  $p_{0 \rightarrow \lambda}(\mathbf{T})$  to achieve in  $\mathbf{T}$  steps a given configuration  $\lambda$  is defined as the ratio of these numbers:

$$p_{0 \rightarrow \lambda}(\mathbf{T}) = \frac{W_{0 \rightarrow \lambda}(\mathbf{T})}{Z_0(\mathbf{T})}$$

Let the initial state be given by the vacuum vector  $\langle 0 |$  and  $o(\mathbf{T}) = J_1^{\mathbf{T}}$ ,  $\mathbf{T} = 0, 1, 2, \dots$  is a discrete time. Then the random process describes the process of creating a Young diagram by gluing boxes to a boundary of Young diagram in a way that at each step we get a Young diagram. A single action action of  $J_1$  on  $\lambda$  glues one box uniformly to any admissible place of the Young diagram of  $\lambda$  (“admissible place” means that a gluing here a box we get a figure which is Young diagram again).  $J_1^{\mathbf{T}}$  glues  $\mathbf{T}$  boxes one by one. Now

$$W_{0 \rightarrow \lambda}(\mathbf{T}) = \langle 0 | J_1^{\mathbf{T}} | \lambda \rangle \quad (286)$$



is an integer which is equal to the number of ways to create a Young diagram of a given shape  $\lambda$  by gluing boxes one by one in a way that each intermediate figure is a Young diagram. It is clear that the weight of the partition is equal to the duration of time:

$$T = |\lambda| \quad (287)$$

otherwise the transition weight vanishes.

The partition function (normalization function) is the sum of all these numbers over final states (Young diagrams)  $\lambda$ :

$$Z_0(\mathbf{T}) = \langle 0 | J_1^{\mathbf{T}} | \Omega_0 \rangle$$

KP tau function (223) evaluated at  $\mathbf{t} = t_1 \mathbf{t}_\infty := (t_1, 0, 0, \dots)$  generates all  $\{W_{0 \rightarrow \lambda}(\mathbf{T}), \mathbf{T} = 0, 1, \dots\}$ :

$$\tau^{KP}(t_1 \mathbf{t}_\infty) = \langle 0 | e^{t_1 J_1} | \lambda \rangle = \sum_{\mathbf{T}} \frac{t_1^{\mathbf{T}}}{\mathbf{T}!} \langle 0 | J_1^{\mathbf{T}} | \lambda \rangle = \sum_{\mathbf{T}} \frac{t_1^{\mathbf{T}}}{\mathbf{T}!} W_{0 \rightarrow \lambda}(\mathbf{T})$$

Let us notice that  $W_{0 \rightarrow \lambda}(\mathbf{T}) = 0$  in case  $\mathbf{T} \neq |\lambda|$  vanish. Taking into account (223) we obtain

$$W_{0 \rightarrow \lambda}(\mathbf{T}) = \mathbf{T}! s_\lambda(\mathbf{t}_\infty) \delta_{\mathbf{T}, |\lambda|} \quad (288)$$

where  $\delta$  is the Kronecker symbol.

IDKP tau function (222) evaluated at  $\mathbf{t} = t_1 \mathbf{t}_\infty := (t_1, 0, 0, \dots)$  generates all  $\{Z(\mathbf{T}) := Z_0(\mathbf{T}), \mathbf{T} = 0, 1, \dots\}$ :

$$\tau_o(t_1 \mathbf{t}_\infty) = \langle 0 | e^{t_1 J_1} | \Omega \rangle = \sum_{\mathbf{T} \geq 0} \frac{t_1^{\mathbf{T}}}{\mathbf{T}!} \sum_{\lambda} W_{0 \rightarrow \lambda}(\mathbf{T}) = \sum_{\mathbf{T} \geq 0} \frac{t_1^{\mathbf{T}}}{\mathbf{T}!} Z_0(\mathbf{T}) \quad (289)$$

On the other hand thanks to the right-hand side of (222) we have  $\tau_o(t_1 \mathbf{t}_\infty) = e^{\frac{1}{2}t_1^2 + t_1}$  which in turn is equal to  $\sum_{\mathbf{T}=0}^{\infty} \frac{t_1^{\mathbf{T}}}{\mathbf{T}!} s_{(\mathbf{T})}(\mathbf{t}_2)$  where  $\mathbf{t}_2 := (1, \frac{1}{2}, 0, 0, \dots)$  and where  $s_{(n)}$  denotes the elementary Schur function known also as  $n$ -th completely symmetric function [18]. Therefore we obtain

$$Z_0(\mathbf{T}) = \mathbf{T}! s_{(\mathbf{T})}(\mathbf{t}_2) = \sum_{n=0}^{\lfloor \frac{\mathbf{T}}{2} \rfloor} \frac{\mathbf{T}! 2^{2n-\mathbf{T}}}{n!(\mathbf{T}-2n)!} \quad (290)$$

Via saddle point method we find that in  $\mathbf{T} \rightarrow \infty$  limit the main contribution in the sum over  $n$  is due to  $n \approx \frac{\mathbf{T}}{2} - \frac{1}{4}\sqrt{\frac{\mathbf{T}}{2}}$  which yields for large  $\mathbf{T}$

$$Z_0(\mathbf{T}) = e^{\frac{\mathbf{T}}{2} \log \mathbf{T} + \frac{\mathbf{T}}{2} (\log 2 - 1) - \frac{1}{4}\sqrt{\mathbf{T}} \log \mathbf{T} + O(\sqrt{\mathbf{T}})}$$

At last we obtain the answer for the probability to achieve a configuration  $\lambda$  in  $\mathbf{T}$  steps:

$$p_{0 \rightarrow \lambda}(\mathbf{T}) = \frac{s_\lambda(\mathbf{t}_\infty)}{s_{(\mathbf{T})}(\mathbf{t}_2)} \delta_{\mathbf{T}, |\lambda|}, \quad \mathbf{t}_\infty := (1, 0, 0, \dots), \quad \mathbf{t}_2 := (1, \frac{1}{2}, 0, 0, \dots) \quad (291)$$

As one can see the probability to achieve the state  $\lambda = (\mathbf{T})$  is given by

$$p_{0 \rightarrow (\mathbf{T})}(\mathbf{T}) = \frac{1}{Z_0(\mathbf{T})}$$

One can ask, given  $\mathbf{T}$ , what is the configuration  $\bar{\lambda}$  which maximizes the number  $W_{0 \rightarrow \lambda}(\mathbf{T})$  and thereby the probability  $p_{0 \rightarrow \lambda}(\mathbf{T})$ . The answer is known [21] and is given by Kerov-Vershik formula for the so-called limit shape of Young diagram, see [71, 72]<sup>10</sup>.

If we modify our random process and admit both creation and elimination of a box we obtain basically the same shape of  $\bar{\lambda}$  however now the weight of this configuration will be less than  $\mathbf{T}$  [21]:

**Example 2.** Starting from the vacuum zero Young diagram, at each time step we either add or remove a box at random in a way that a figure we obtain during at each time step is a Young diagram,

<sup>10</sup>The reason of this coincidence is the following. The Vershik-Kerov limit shape YD maximizes Plancherel measure on partitions, and Plancherel measure is basically  $(s_\lambda(\mathbf{t}_\infty))^2$ .

see the figure below. This model is equivalent to a model of random turn walk suggested in [70] where initial configuration of walkers is the step function.



1. Random turn walk of particles on a Maya diagram.

2. Random adding/removing a box to a Young diagram related to the up/downward hops of particles on Maya diagram. At unit time instant either a box has to be added at any of vacant places marked by star, or a box marked by x has to be removed.

This model describes hard core particles ("walkers", "hard core" means that two particles can not occupy the same site) situated at the sites of 1D lattice. The model implies that at each tick of clock one chosen at random particle hops either to the left or to the right. In our picture particles are fermions which we placed on the vertical lattice. The step function is the vacuum state,  $\langle 0|$ , describing the Dirac sea where all sites downward to the sea level are occupied. We count particles from the top. Excitations may be described by partitions  $\lambda = (\lambda_1, \lambda_2, \dots)$  where  $\lambda_1$  describes the upward shift of the up-most particles (the particle number one) with respect to its original position in the Dirac seas, the shift of the particle number  $i$  is equal to  $\lambda_i$ . It is clear that  $\lambda_1 \geq \lambda_2 \geq \dots$ . Such configuration is denoted by  $\langle \lambda|$ . Each upward step of a particle from a configuration  $\lambda$  may be described as gluing a box to the Young diagram of  $\lambda$ , while each downward step is described as removing one box from the Young diagram of  $\lambda$ .

The number of paths of a length  $T$  which start at the vacuum configuration and end at a given configuration  $\lambda$  divided by the number of all paths of the length  $T$  which start at the vacuum configuration defines the transition probability  $p_{0 \rightarrow \lambda}(T)$ .

Now the number of ways to achieve a given configuration  $\lambda$  during a lapse of time  $T$  starting from the vacuum (step function) configuration is

$$W_{0 \rightarrow \lambda}(T) = \langle 0|(J_1 + J_{-1})^T|\lambda\rangle \quad (292)$$

where  $T$  is not necessarily equal to  $|\lambda|$ .

The number of ways to achieve any configuration in  $T$  time steps starting from the vacuum configuration is

$$Z_0(T) = \langle 0|(J_1 + J_{-1})^T|\Omega_0\rangle$$

A usage of Baker-Campbell-Hausdorff formula  $e^{tJ_1 + tJ_{-1}} = e^{\frac{1}{2}t^2} e^{tJ_{-1}} e^{tJ_1}$  may be considered as an advantage of the fermionic approach. After some algebra we obtain [21]

**Proposition 15.** *We have*

$$W_{0 \rightarrow \lambda}(T) = T! \cdot s_\lambda(\mathbf{t}_\infty) \delta(T, |\lambda|), \quad \delta(T, |\lambda|) := 2^{\frac{|\lambda|-T}{2}} \frac{1}{(\frac{T-|\lambda|}{2})!} \quad (293)$$

where  $\delta(T, |\lambda|)$  replaces the Kronecker symbol  $\delta_{T, |\lambda|}$ . In (293)  $T - |\lambda|$  is an even number otherwise  $W_{0 \rightarrow \lambda}(T)$  vanishes.

The following IDKP tau function generates partition functions

$$\langle 0 | e^{tJ_1 + tJ_{-1}} | \Omega \rangle = \sum_{\mathbf{T} \geq 0} \frac{t^{\mathbf{T}}}{\mathbf{T}!} \sum_{\lambda} W_{0 \rightarrow \lambda}(\mathbf{T}) = \sum_{\mathbf{T} \geq 0} \frac{t^{\mathbf{T}}}{\mathbf{T}!} Z(\mathbf{T}) \quad (294)$$

On the other hand

$$\langle 0 | e^{tJ_1 + tJ_{-1}} | \Omega \rangle = e^{\frac{1}{2}t^2} \langle 0 | e^{tJ_1} | \Omega \rangle = e^{t^2 + t} = \sum_{\mathbf{T}=0}^{\infty} t^{\mathbf{T}} s_{(\mathbf{T})}(\mathbf{t}'_2) \quad (295)$$

where  $\mathbf{t}'_2 := (1, 1, 0, 0, \dots)$ . Therefore we get

**Proposition 16.** *The number of paths of the length  $T$  which start at the vacuum configuration is given by*

$$Z_0(\mathbf{T}) = \mathbf{T}! s_{(\mathbf{T})}(\mathbf{t}'_2), \quad \mathbf{t}'_2 := (1, 1, 0, 0, \dots) \quad (296)$$

where  $s_{(\mathbf{T})}(\mathbf{t}'_2)$  is the elementary Schur function related to the partition  $(\mathbf{T})$ :

$$s_{(\mathbf{T})}(\mathbf{t}'_2) = \sum_{n=0}^{\lfloor \frac{T}{2} \rfloor} \frac{1}{n!(T-2n)!} \quad (297)$$

The number of paths of a length  $T$  which start at the vacuum configuration and end at a given configuration  $\lambda$  divided by the number of all paths of the length  $T$  which start at the vacuum configuration is given by

$$p_{0 \rightarrow \lambda}(\mathbf{T}) = \frac{s_{\lambda}(\mathbf{t}_1)}{s_{(\mathbf{T})}(\mathbf{t}'_2)} \delta(\mathbf{T}, |\lambda|), \quad \mathbf{t}_1 := (1, 0, 0, \dots), \quad \mathbf{t}'_2 := (1, 1, 0, 0, 0, \dots) \quad (298)$$

As one can see

$$p_{0 \rightarrow (\mathbf{T})}(\mathbf{T}) = \frac{1}{Z_0(\mathbf{T})}$$

For large  $\mathbf{T}$  one may apply the saddle point method to evaluate the sum (297). The saddle point is related to  $n = \frac{T}{2} - \frac{1}{2} \sqrt{\frac{T}{2}} + O(1)$ . This yields

**Proposition 17.** *In large  $T$  limit we obtain*

$$Z_0(\mathbf{T}) = e^{\frac{T}{2} \log T + \frac{T}{2} \log \frac{2}{e} + O(\sqrt{T})} \quad (299)$$

$$p_{0 \rightarrow \lambda}(\mathbf{T}) = s_{\lambda}(\mathbf{t}_1) e^{-\frac{T}{2} \log 2 + O(\sqrt{T})} \quad (300)$$

**'One dimensional dimer' target configurations** At last let us focus on the following problem. Let us evaluate the number of paths of a given length  $\mathbf{T}$  which end on a configuration related to a fat partition  $\lambda \cup \lambda$  :

$$N_{\text{FP}}(\mathbf{T}) := \sum_{\lambda \in \mathcal{P}} W_{0 \rightarrow \lambda \cup \lambda}(\mathbf{T}) \quad (301)$$

The number  $N_{\text{FP}}(\mathbf{T})$  vanishes in case  $\mathbf{T}$  is odd. Let us recall that each configuration  $\lambda \cup \lambda$  describes a configuration of pairs of particles ('one dimensional dimers').

**Proposition 18.**

$$N_{\text{FP}}(\mathbf{T}) = \begin{cases} 2^{\frac{1}{2}T} (T-1)!! & \text{iff } T \text{ is even} \\ 0 & \text{iff } T \text{ is odd} \end{cases} \quad (302)$$

Indeed, on the one hand

$$\sum_{\lambda \in \mathcal{P}} \langle 0 | e^{tJ_1 + tJ_{-1}} | \lambda \cup \lambda \rangle = \sum_{\mathbf{T} \geq 0} \frac{t^{\mathbf{T}}}{\mathbf{T}!} \sum_{\lambda} W_{0 \rightarrow \lambda \cup \lambda}(\mathbf{T}) = \sum_{\mathbf{T} \geq 0} \frac{t^{\mathbf{T}}}{\mathbf{T}!} N_{\text{FP}}(\mathbf{T}) \quad (303)$$

On the other hand (see (82))

$$\sum_{\lambda \in \mathcal{P}} \langle 0 | e^{tJ_1 + tJ_{-1}} | \lambda \cup \lambda \rangle = e^{\frac{1}{2}t^2} \sum_{\lambda \in \mathcal{P}} \langle 0 | e^{tJ_1} | \lambda \cup \lambda \rangle = e^{t^2} = \sum_{\mathbf{T}=0,2,4,\dots} \frac{t^{\mathbf{T}}}{(\frac{1}{2}\mathbf{T})!} \quad (304)$$

## Acknowledgements

We are grateful to John Harnad, Johan van de Leur, Vladimir Osipov and and most of all to Eugene Kanzieper for discussions of the topic. One of the authors (A.O) thanks E. Kanzieper and Holon Technology Institute for hospitality (July 2008) where a part of this work (a main part of the section "Asymmetric two-matrix ensemble" in [1]) was done. We thank Andrei Mironov for the reference [40]. The work was supported by RFBR grants .... and by Japanese-RFBR grant 10-01-92104 JF and also by RAS Program "Fundamental Methods in Nonlinear Dynamics". This work is also partly supported by Grant-in-Aid for Scientific Research No. 22540186 from the Japan Society for the Promotion of Science and by the Bilateral Joint Project "Integrable Systems, Random Matrices, Algebraic Geometry and Geometric Invariants" (2010–2011) of the Japan Society for the Promotion of Science and the Russian Foundation for Basic Research.

## References

- [1] A. Yu. Orlov, T. Shiota and K. Takasaki, "Pfaffian structures and certain solutions to BKP hierarchies II. Multi-integrals"
- [2] M. Adler and P. van Moerbeke, "Symmetric random matrices and the Pfaff lattice", arXiv:solv-int/9903009v1
- [3] M. Adler, P. van Moerbeke and T. Shiota, "Pfaff  $\tau$ -functions", arXiv:nlin/9909010
- [4] E. Date, M. Jimbo, M. Kashiwara and T. Miwa, "Transformation groups for soliton equations, IV A new hierarchy of soliton equations of KP-type", *Physica* **4D** (1999982) 343-365
- [5] E. Date, M. Jimbo, M. Kashiwara and T. Miwa, "Operator Approach to the Kadomtsev-Petviashvili Equation. Transformation Group for Soliton Equations III". *J. Phys. Soc. Japan* **50** (1981), 3806-3812
- [6] M. Jimbo and T. Miwa, "Solitons and Infinite Dimensional Lie Algebras", *Publ. RIMS Kyoto Univ.* **19**, 943–1001 (1983).
- [7] M. Sato and Y. Sato(Mori), *RIMS Kokyuroku* **388**, Kyoto Univ. (1980) 183, **414** (1981) 181
- [8] C. Krattenthaler, A.J. Guttman, and X.G. Viennot, "Vicious Walkers, Friendly Walkers and Young Tableaux II: With a Wall", ArXiv cond-mat/0006367 v 2
- [9] , C. Krattenthaler, "Identities for Classical Group Characters of Nearly Rectangular Shape", arXiv:math/9808118 (1998); M. Ciucu, C. Krattenthaler, "A factorization theorem for classical group characters, with applications to plane partitions and rhombus tilings", in: *Advances in Combinatorial Mathematics: Proceedings of the Waterloo Workshop in Computer Algebra 2008*, I. Kotsireas, E. Zima (eds.), Springer-Verlag, 2010, pp. 39-60; arXiv:0812.1251
- [10] Y. You, "Polynomial solutions of the BKP hierarchy and projective representations of symmetric groups", in *Infinite-dimensional Lie algebras and groups*, pp. 449-464, Adv. Ser. Math. Phys., **7**. World Science Publishing, Teaneck, New Jersey
- [11] J.J.C. Nimmo, "Hall-Littlewood symmetric functions and the BKP equation", *J. Physics A*, **23**, 751-760
- [12] R.A.Proctor, Odd symplectic groups, *Invent. Math.* **92** (1988), 307-332
- [13] V. Kac and J. van de Leur, "The Geometry of Spinors and the Multicomponent BKP and DKP Hierarchies", CRM Proceedings and Lecture Notes **14** (1998) 159-202
- [14] J.W. van de Leur, "Matrix Integrals and Geometry of Spinors", *J. of Nonlinear Math. Phys.* **8**, 288-311 (2001)
- [15] J. Harnad, private communication.

- [16] J. Harnad and S. Snider, “Isotropic geometry and twistors in higher dimensions. I. The generalized Klein correspondence and spinor flags in even dimensions”, *J. Math. Phys.* **33** No. 9 pp.3197-3208 (1992); J. Harnad and S. Snider, “Isotropic geometry and twistors in higher dimensions. II. Odd dimensions, reality conditions, and twistor superspaces”, *J. Math. Phys.* **36** No. 4 pp. 1945-1970 (1995)
- [17] M. L. Mehta, *Random Matrices*, 3rd edition (Elsevier, Academic, San Diego CA, 2004).
- [18] I.G. Macdonald, *Symmetric Functions and Hall Polynomials*, Clarendon Press, Oxford, (1995).
- [19] A. Pandey and M.L. Mehta, “Gaussian ensembles of random Hermitian matrices intermediate between orthogonal and unitary ones, *Comm.Math.Phys.* **87**, (1983) 449-468; M.L. Mehta and A. Pandey, “Spacing distribution for some Gaussian ensembles of Hermitian matrices”, *J.Phys. A* **16** (1983) L601-606; M.L. Mehta and A. Pandey, “On some Gaussian ensembles of Hermitian matrices”, *J.Phys. A* **16** (1983) 2655-2684
- [20] J. Harnad and A.Yu. Orlov, “Fermionic construction of partition functions for two-matrix models and perturbative Schur function expansions”, *J. Phys. A* **39** 8783-8810 (2006), arXiv math-ph/0512056 v3
- [21] J. Harnad and A.Yu. Orlov, “Fermionic tau function and random processes, *Physica D: Nonlinear Phenomena* **235**, Issues 1-2, (2007) 168-206
- [22] J. Harnad and A.Yu. Orlov, “Convolution symmetries of integrable hierarchies, matrix models and tau functions”, preprint CRM-3272 (2008), arXiv:0901.0323 [math-ph];
- [23] J. Harnad and A.Yu. Orlov, “Convolution symmetry flows and integrable hierarchies”, preprint (2008)
- [24] K. Takasaki, “Auxiliary linear problem, difference Fay identities and dispersionless limit of Pfaff-Toda hierarchy”, nlin/09083569
- [25] J. Harnad and A.Yu. Orlov, “Scalar products of symmetric functions and matrix integrals”, *Theor. Math. Phys.* **137**, 1676–1690 (2003); arXiv: nlin.SI/0211051
- [26] J.P. Harnad, J.W. van de Leur and A.Yu.Orlov, “Multiple sums and integrals as neutral BKP tau functions”, arXiv:math-ph/11014216
- [27] A.Yu. Orlov, “Hypergeometric Functions as Infinite-Soliton Tau function” *Theoretical and Mathematical Physics* **146**(2): 183-206 (2006); “Hypergeometric  $\tau$ -functions  $\tau(\mathbf{t}, T, \mathbf{t}^*)$  as  $\infty$ -soliton  $\tau$ -function in  $T$  variables”, *nonlin.SI/0305001*
- [28] T. Nakatsu and K. Takasaki, “Melting Crystal, Quantum Torus and Toda Hierarchy”, arXiv hep-th/07105339
- [29] A.Yu. Orlov and D.M. Scherbin “Fermionic representation for basic hypergeometric functions related to Schur polynomials” nlin.SI/0001001
- [30] A.Yu. Orlov and D.M. Scherbin, “Multivariate hypergeometric functions as tau functions of Toda lattice and Kadomtsev-Petviashvili equation”, *Physics D* **152-153** pp 51-56 (2001)
- [31] A.Yu. Orlov, “Hypergeometric functions related to Schur Q-polynomials and BKP equation”, *Theoretical and Mathematical Physics*, **137** (2): 1573-1588 (2003), ArXiv:math-ph/0302011
- [32] A.Yu. Orlov and T. Shiota, “Schur function expansion for normal matrix model and associated discrete matrix models”, *Phys. Lett. A* **343**, 384-396 (2005); (more completed version in math-ph/0501017)
- [33] S. C. Milne, “Summation theorems for basic hypergeometric series of Schur function argument”. In: Gonchar, A.A. and Saff, E.B. (eds) *Progress in Approximation Theory*, pp. 51-77. Springer-Verlag, New-York, 1992. See also: Vilenkin, N.Ya. and Klimyk, A.U

- [34] N. Ya. Vilenkin, and A. U. Klimyk, *Representation of Lie groups and Special Functions, Recent Advances*, Kluwer Academic Publishers, 1995
- [35] K. I. Gross, and D. S. Richards, *Special functions of matrix arguments. I: Algebraic induction, zonal polynomials, and hypergeometric functions*, *Transactions Amer Math Soc* **301** (1987) 781-811
- [36] V.A. Osipov, H.-J. Sommers and K. Zyczkowski, “Random Bures mixed states and the distribution of their purity” , arXiv cond-math.stat-mech/09095094 v1
- [37] E.T. Whittaker and G.N. Watson, *A course of modern analysis*. CUP, Cambridge, 2nd edition, 1965
- [38] G. Gaspar and M. Rahman, Basic hypergeometric series, *Encyclopedia of Mathematics and its Applications*, Vol. 96, Second Edition, Cambridge Iniversity Press, Cambridge, 2004
- [39] N.Ya. Vilenkin and A.U. Klimyk *Representation of Lie Groups and Special Functions. Volume 3: Classical and Quantum Groups and Special Functions*, Kluwer Academic Publishers, 1992
- [40] S. Kharchev, A. Marshakov, A. Mironov and A. Morozov, ”Generalized Kazakov-Migdal-Kontsevich Model: group theory aspects”, arXiv/hep-th/9312210
- [41] A. Morozov, “Integrablity and Matrix Models”, *Phys. Usp.* **37** 1 (1997), arXiv/hep-th/9303139
- [42] A. K. Pogrebkov and V. N. Sushko, “ Quantization of the  $(\sin\psi)_2$  interaction in terms of fermion variables”, Translated from *Teoreticheskaya i Matematicheskaya Fizika*, Vol. 24, No. 3, pp.425-429, September, 1975. Original article submitted May 15, 1975
- [43] V.E. Zakharov, S.V. Manakov, S.P. Novikov (ed), L. Pitaevsky, *Theory of Solitons*, Nauka, Moscow, 1979
- [44] K. Takasaki, “Initial value problem for the Toda lattice hierarchy”, *Adv. Stud. Pure Math.* **4**, 139-163 (1984).
- [45] T. Takebe, “Representation Theoretical Meaning of Initial Value Problem for the Toda Lattice Hierarchy I, *LMP* **21** (1991) 77–84; T. Takebe, “Representation Theoretical Meaning of Initial Value Problem for the Toda Lattice Hierarchy II”, *Publ. RIMS, Kyoto Univ.* **27** (1991), 491–503
- [46] T. Nakatsu, K. Takasaki, and S. Tsujimaru, “Quantum and Classical Aspects of Deformed  $c = 1$  Strings”, *Nucl. Phys. B* **443** (1995) 155; *hep-th/9501038*
- [47] N. Nekrasov and A. Okounkov, “Seiberg-Witten Theory and Random Partitions”, arxiv: hep-th/0306238
- [48] A. A. Migdal, *Zh. Eksp. Teor. Fiz* **69** (1975) 810 (*Sov Phys JETP* 42 413)
- [49] K. Saaidi, and M. Khorrami, “Large- $N$  limit of the nonlocal 2D Yang-Mills and generalized Yang-Mills theories on a cilinder”, *hep-th/0304204*
- [50] A. Okounkov, “The uses of random partitions”, Section 1.4.4, arxiv: math-ph/0309015
- [51] A. Okounkov, “Toda equations for Hurwitz numbers”, arXiv: *AG/00044128*
- [52] A. Okounkov, and N. Reshetikhin, “Random skew plane partitions and the Pearcey process”, arXiv:math/0503508
- [53] A. Okounkov, and R. Pandharipande, “The equivariant Gromov-Witten theory on  $P^1$ ”, arXiv: *AG/0207233*
- [54] W.-P. Li, Z. Qin and W. Wang, “Hilbert Schemes, Integrable Hierarchies and Gromov-Witten Theory”, arXiv: *AG/0302211*
- [55] I.P. Goulden, D.M. Jackson, “The KP hierarchy, branched covers, and triangulations”, arXiv:0803.3980

- [56] A.G. Abanov, E. Bettelheim, and P. Wiegmann, “Nonlinear dynamics of quantum systems and soliton theory”, *J. Phys. A* **40**, F193-F207 (2007).
- [57] A. Mironov, A. Morozov and S. Natanzon, “Complete Set of Cut-and-Join Operators in Hurwitz-Kontsevich Theory”, arXiv/hep-th/0904.4227
- [58] M. Katori and N. Komatsuda, “Moments of vicious walkers and Mobius graph expensions”, 0212500
- [59] T. Nagao, “Pfaffian Expressions for Random Matrix Correlation Functions”, *J. Stat. Phys.* **129** (2007) 1137; arxiv: math-phys/07082036
- [60] M. Katori and H. Tanemura, “Scaling limit of vicious walks and two-matrix model”, *Phys. Rev. E* **66** (2002) 011105; arxiv:cond-math/02035449
- [61] O. Foda, M. Wheeler and M. Zuparic, “On free fermions and plane partitions”, arXiv, math-ph/08082737
- [62] O. Foda, M. Wheeler and M. Zuparic, “XXZ scalar products and KP”, Nucl. Phys. B820 [FS] (2009), 649663.
- [63] P. Zinn-Justin, “Six-vertex, loop and tiling models: Integrability and combinatorics”, arXiv:0901.0665.
- [64] K. Takasaki, “KP and Toda tau functions in Bethe ansatz”, arXiv:1003.3071
- [65] A. Alexandrov, V. Kazakov, S.n Leurent, Z. Tsuboi, A. Zabrodin, “Classical tau-function for quantum spin chains”, arXiv:1112.3310
- [66] H. Braden, Correlation functions in the 2D Ising model, private communication (2003)
- [67] S. Matsumoto, “Alpha-Pfaffian, pfaffian point process and shifted Schur measure Linear Algebra and its Applications”, 403 (2005) 369–398, arXiv:math/0411277; S. Matsumoto, “Correlation functions of the shifted Schur measure”, *J. Math. Soc. Japan*, vol.57, no. 3 (2005), 619–637, arXiv:math/0312373
- [68] M. Vuletic, “Shifted Schur Process and Asymptotics of Large Random Strict Plane Partitions”, arXiv:math-ph/0702068v1
- [69] J. van de Leur and A.Yu. Orlov, “Random turn walk on a half line with creation of particles at the origin”, *Physics Letters A* 31, July 2009, Pages 2675-2681; arxiv: math-ph/0801.0066
- [70] M. Fisher, “Walks, walls, wetting and melting”, *J. Stat. Phys.* **34** 667-729 (1984)
- [71] A.M. Vershik and S.V. Kerov, “Asymptotics of the Plancherel measure of the symmetric group and the limit form of Young tableaux”, *Soviet Mathematics Doklady* **18** (1977) 527-531
- [72] A.M. Vershik and S.V. Kerov, “Asymptotics of the largest and the typical dimensions of the irreducible representations of a symmetric group”, *Funktsional’nyi Analiz i Ego Prilozheniya*, Vol. 19, No. 1, pp. 25-36 (1985)
- [73] A. Okounkov, “Infinite wedge and random partitions”, *Selecta Math. (N.S.)* **7** (2001), 57-81, arXiv:math/9907127
- [74] A. Borodin and E. Strahov, “Correlation Kernal for Discrete Symplectic and Orthogonal Ensembles”, arXiv: math-phys/07121693
- [75] A.R. Its, C. Tracy and H. Widom, “Random Words, Toeplitz Determinants and Integrable Systems II. Advances in Non-linear Mathematics and Science”, *Phys. D*, **152-153** (2001) 199-224
- [76] J. Baik E and Rains, “The asymptotics of monotone subsequence of involutions”, arxiv: math.CO/9905084
- [77] J. Baik E and Rains, “Algebraic aspects of increasing subsequences”, arxiv: math.CO/9905083

- [78] K. Johanssen, “Discrete Orthogonal Polynomial Ensembles and the Plancherel Measure”, *Ann. of Math.* **153** (2001) 199-224
- [79] J. Baik, P. Deift and K. Johansson, “On the Distribution of the Length of the Longest Increasing Subsequence of Random Permutations”, *J. Amer. Math. Soc.*, **12**, No 4 (1999) 1119-1178
- [80] J. Baik, “Random vicious walk and random matrices”, *Comm. Pure Appl. Math.*, 53(11) 11 85-1410 (2000.)
- [81] T.H. Baker and P.J. Forrester, “Random walks and random fixed point free involutions”, arXiv: math.CO/0107128
- [82] M. Douglas and V. Kazakov, “Large  $N$  Phase Transition in Continuum  $QCD_2$ ”, arxiv: hep-th/9305047
- [83] V. Kazakov and T. Wynter, “Large  $N$  Phase Transition in the Heat Kernel on the  $U(N)$  Group”, *Nucl. Phys.* **B440** 407-420; arxiv: hep-th/9410087
- [84] V. Kazakov, M. Staudacher and T. Wynter, “Character Expansion Methods for Matrix Models of Dually Weighted Graphs”, *Comm. Math. Phys.* **177** (1996) 451-468; arxiv: hep-th/9502132
- [85] I.M. Loutsenko and V. Spiridonov, “Soliton Solutions of Integrable Hierarchies and Coulomb Plasmas”, *J. Stat. Phys.*, **99**, 751, (2000).
- [86] A. Mironov, A. Morozov and G. Semenoff, “Unitary Matrix Integrals in the Framework of the Generalized Kontsevich Model”, *Intern Journ Mod Phys A* **11** (1996) 5031-5080
- [87] C.A. Tracy and H. Widom, “A Limit Theorem for Shifted Schur Measures”, arxiv: math.PR/0210255
- [88] M. Adler and P. van Moerbeke “Virasoro action on Schur function expansions, skew Young tableaux and random walks”
- [89] P.J. Forrester, “Log Gases and Random Matrices”, book in preparation <http://www.ms.unimelb.edu.au/~matpjf.matpjf.html>
- [90] J.W. van de Leur, A. Yu. Orlov, T. Shiota, “CKP Hierarchy, Bosonic Tau Function and Bosonization Formulae”, arXiv: math-ph/11020087
- [91] M. Ishikawa, H. Kawamuko, S. Okada, A Pfaffian–Hafnian Analogue of Borchardt’s Identity, *The Electronic Journal of Combinatorics* **12** (2009), N 9 ; math.CO/048364.
- [92] J.R. Stembridge, “Shifted Tableaux and the Projective Representations of Symmetric Groups”, *Advances in Mathematics* **74**, 87-134 (1989)
- [93] J. Stembridge, “Nonintersecting paths, pfaffians and plane partitions”, *Adv. Math.* **83** (1990) 96-131
- [94] M. Ishikawa and M. Wakayama, “Applications of minor summation formula III, Plucker relations, lattice path and Pfaffian identities”, *Journal of Combinatorial Theory, Series A* **113** (2006) 113-155
- [95] B.A. Hsiao and O.S. Kravchenko, “A central extension of the algebra of pseudodifferential symbols”, *Funct. Anal. Appl.* **25** (1991) 83.
- [96] A. Yu. Orlov, “Tau Functions and Matrix Integrals”, arXiv: math-ph/0210012
- [97] A. Yu. Orlov, “New Solvable Matrix Integrals”, *International Journal of Mathematical Physics A*, Vol. 19, Supplement (2004) 276-293, ArXiv: nlin/0209064



## A Appendices

### A.1 Pfaffians. Partitions. Schur functions

**(A) Pfaffians.** We need the notion of Pfaffian. If  $A$  an anti-symmetric matrix of an odd order its determinant vanishes. For even order, say  $k$ , the following multilinear form in  $A_{ij}, i < j \leq k$

$$\text{Pf}[A] := \sum_{\sigma} \text{sgn}(\sigma) A_{\sigma(1),\sigma(2)} A_{\sigma(3),\sigma(4)} \cdots A_{\sigma(k-1),\sigma(k)} \quad (305)$$

where sum runs over all permutation restricted by

$$\sigma : \sigma(2i-1) < \sigma(2i), \quad \sigma(1) < \sigma(3) < \cdots < \sigma(k-1), \quad (306)$$

coincides with the square root of  $\det A$  and is called the Pfaffian of  $A$ , see, for instance [17]. As one can see the Pfaffian contains  $1 \cdot 3 \cdot 5 \cdots (k-1) =: (k-1)!!$  terms.

The following equality is known as Schur identity

$$\text{Pf} \left( \left( \frac{x_i - x_j}{x_i + x_j} \right)_{1 \leq i, j \leq 2n} \right) = \Delta_{2n}^*(x) \quad (307)$$

where

$$\Delta_k^*(x) := \prod_{1 \leq i < j \leq k} \frac{x_i - x_j}{x_i + x_j} \quad (308)$$

Let us mark that a special case of this relation is obtained if  $x_{2n}$  vanishes. In this case we write

$$\text{Pf}(A) = \Delta_{2n-1}^*(x) \quad (309)$$

where  $A$  is an antisymmetric  $2n \times 2n$  matrix defined by

$$A_{ij} = \begin{cases} \frac{x_i - x_j}{x_i + x_j} & \text{if } 1 < i < j < 2n \\ 1 & \text{if } i < j = 2n, \end{cases} \quad (310)$$

**Hafnians** The *Hafnian* of a symmetric matrix  $A$  of even order  $N = 2n$  is defined as

$$\text{Hf}(A) := \sum_{\sigma} A_{\sigma(1),\sigma(2)} A_{\sigma(3),\sigma(4)} \cdots A_{\sigma(2n-1),\sigma(2n)} \quad (311)$$

where sum runs over all permutation restricted by

$$\sigma : \sigma(2i-1) < \sigma(2i), \quad \sigma(1) < \sigma(3) < \cdots < \sigma(2n-1), \quad (312)$$

As one can see the this sum contains  $1 \cdot 3 \cdot 5 \cdots (2N-1) =: (2N-1)!!$  terms.

*Remark 22.* Let us note that entries on the diagonal of the matrix  $A$  does not contribute the sum (311).

The following equality was found in [91]

$$\text{Pf} \left( \left( \frac{x_i - x_j}{(x_i + x_j)^2} \right)_{1 \leq i, j \leq 2n} \right) = \prod_{1 \leq i < j \leq 2n} \frac{x_i - x_j}{x_i + x_j} \text{Hf} \left( \left( \frac{1}{x_i + x_j} \right)_{1 \leq i, j \leq 2n} \right) \quad (313)$$

Another proof of this relation was presented in [90]. Let us mark that a special case of this relation is obtained if  $x_{2n}$  vanishes. In this case we write

$$\text{Pf}(B) = \Delta_{2n-1}^*(x) \text{Hf}(C) =: \Delta_N^{**}(x) \quad (314)$$

where  $B$  and  $C$  are respectively antisymmetric and symmetric  $2n \times 2n$  matrices whose relevant entries (see Remark 22) are given by

$$B_{ij} = \begin{cases} \frac{x_i - x_j}{(x_i + x_j)^2} & \text{if } 1 \leq i < j < 2n \\ \frac{1}{x_i} & \text{if } i < j = 2n, \end{cases} \quad C_{ij} = \begin{cases} \frac{1}{x_i + x_j} & \text{if } 1 \leq i < j < 2n \\ \frac{1}{x_i} & \text{if } i < j = 2n, \end{cases} \quad (315)$$

**(B) Partitions.** Polynomial functions in many variables, like the Schur functions, are parameterized by partitions.

Let us remind that a partition of certain number  $n$  is an ordered set of integers  $\lambda = (\lambda_1, \dots, \lambda_i)$  where  $\lambda_1 \geq \dots \geq \lambda_i \geq 0$  such that  $n = \sum_{k=1}^i \lambda_k$ . Then,  $n$  is called the weight of  $\lambda$  and commonly denoted by  $|\lambda|$ , see [18]. Integers  $\lambda_k$  are called parts of the partition  $\lambda$ . The number of non-vanishing parts of  $\lambda$  is called the length of  $\lambda$  and will be denoted by  $\ell(\lambda)$ .

Almost everywhere throughout the paper we will denote partitions by Greek characters.

Strictly ordered sets  $\alpha = (\alpha_1, \dots, \alpha_i)$ ,  $\alpha_1 > \dots > \alpha_i \geq 0$  are called the strict partitions, see [18]. In this paper we write  $\{\alpha\}$  to denote strictly ordered sets  $\alpha_1 > \dots > \alpha_i$  where numbers  $\alpha_k$  are not necessarily positive.

We basically use notations adopted in [18].

**Young diagrams** The (*Young*) *diagram* of a partition  $\lambda$  is defined as the set of points (or nodes)  $(i, j) \in \mathbb{Z}^2$ , such that  $1 \leq j \leq \lambda_i$ . Thus, it is a subset of a rectangular array with  $\ell(\lambda)$  rows and  $\lambda_1$  columns. We denote the diagram of  $\lambda$  by the same symbol  $\lambda$ . For example,

$$\begin{array}{|c|c|c|} \hline & & \\ \hline & & \\ \hline & & \\ \hline & & \\ \hline \end{array} \quad (316)$$

is the diagram of  $(3, 3, 1)$ . The weight of this partition is 7, the length is equal to 3.

The partition whose diagram is obtained by the transposition of the diagram  $\lambda$  with respect to the main diagonal is called the conjugated partition and denoted by  $\lambda^t$ .

In the Frobenius notations (see [18]) we write  $\lambda = (\alpha_1, \dots, \alpha_k | \beta_1, \dots, \beta_k)$  or just  $\lambda = (\alpha | \beta)$ . For instance the partition (316) is written as  $(2, 1 | 2, 0)$ . The partition  $\lambda^{tr} := (\beta | \alpha)$  is called transposed to  $\lambda = (\alpha | \beta)$ .

**Hook polynomials, Pochhammer symbols** The *product of hook lengths*  $H_\lambda$  is defined as

$$H_\lambda = \prod_{i,j \in \lambda} h_{ij}, \quad h_{ij} = \lambda_i - i + \lambda_j^t - j + 1, \quad (317)$$

where the product ranges over all nodes of the diagram of the partition  $\lambda$ .

Given number  $q$ , the so-called hook polynomial  $H_\lambda(q)$  is defined as:

$$H_\lambda(q) = \prod_{i,j \in \lambda} (1 - q^{h_{ij}}), \quad h_{ij} = \lambda_i - i + \lambda_j^t - j + 1 \quad (318)$$

In what follows, we also need notations:

$$n(\lambda) := \sum_{i=1}^k (i-1)\lambda_i, \quad (319)$$

$$(a)_\lambda := (a)_{\lambda_1} (a-1)_{\lambda_2} \dots (a-k+1)_{\lambda_k}, \quad (a)_m := \frac{\Gamma(a+m)}{\Gamma(a)}, \quad (320)$$

$$(q^a; q)_\lambda := (q^a; q)_{\lambda_1} (q^{a-1}; q)_{\lambda_2} \dots (q^{a-k+1}; q)_{\lambda_k}, \quad (q^a; q)_m := (1 - q^a) \dots (1 - q^{a+m-1}), \quad (321)$$

where  $k = \ell(\lambda)$ . We set  $(a)_0 = 1$  and  $(q^a; q)_0 = 1$ .

Useful relations are

$$(a)_\lambda = c_k(a) \prod_{i=1}^k (a)_{h_i+1-k}, \quad (q^a; q)_\lambda = c_k(a, q) \prod_{i=1}^k (q^a; q)_{h_i+1-k} \quad (322)$$

where

$$c_k(a) = \prod_{i=1}^k (a-i)^{k-i}, \quad c_k(a, q) = \prod_{i=1}^k (1 - q^{a-i})^{k-i}, \quad h_i = \lambda_i - i + k \quad (323)$$

**Schur functions.** We now consider a semi-infinite set of variables  $\mathbf{t} = (t_1, t_2, t_3, \dots)$ . Given partition  $\lambda$ , the Schur function  $s_\lambda(\mathbf{t})$  is defined by

$$s_\lambda(\mathbf{t}) = \det(h_{\lambda_i - i + j}(\mathbf{t}))_{1 \leq i, j \leq \ell(\lambda)}, \quad \text{where} \quad \sum_{k=0}^{\infty} z^k h_k(\mathbf{t}) = \exp \sum_{m=1}^{\infty} z^m t_m, \quad (324)$$

and, for  $k < 0$ , we put  $h_k = 0$ . The  $h_k(\mathbf{t})$  is called the elementary Schur function.

There is another definition of the Schur function; it is the following symmetric function in the different variables  $x := x^{(n)} := (x_1, \dots, x_n)$ , where  $n \geq \ell(\lambda)$ :

$$\underline{s}_\lambda(x) = \frac{\det(x_i^{\lambda_j - j + n})_{1 \leq i, j \leq n}}{\det(x_i^{n-j})_{1 \leq i, j \leq n}}; \quad (325)$$

for the zero partition one puts  $\underline{s}_0(x) = 1$ . If

$$\mathbf{t} = \mathbf{t}(x^{(n)}) = (t_1(x^{(n)}), t_2(x^{(n)}), \dots), \quad t_m(x^{(n)}) = \frac{1}{m} \sum_{i=1}^n x_i^m,$$

then definitions (324) and (325) are equivalent [18]:

$$s_\lambda(\mathbf{t}(x^{(n)})) = \underline{s}_\lambda(x^{(n)}). \quad (326)$$

*Remark 23.* From definition (324) it follows that  $s_\lambda(\mathbf{t}(x^{(n)})) = 0$  if  $\ell(\lambda) > n$ .

The Schur functions  $\underline{s}_\lambda(x_1, \dots, x_n)$ , where  $\ell(\lambda) \leq n$ , form a basis in the space of symmetric functions in  $n$  variables. We use the underline in  $\underline{s}_\lambda$  only to distinguish the two definitions. If an  $n \times n$  matrix  $X$  has eigenvalues  $x_1, \dots, x_n$ , we may denote  $\underline{s}_\lambda(x_1, \dots, x_n)$  by  $s_\lambda(X)$ , without underline, since in this paper the Schur function with uppercase argument is used only in this sense.

One of the wonderful results of Kyoto school is the formula

$$s_\lambda(\mathbf{t}) = \langle 0 | \Gamma(\mathbf{t}) | \lambda \rangle \quad (327)$$

which may be obtained by the direct calculation using (324) and  $\Gamma(\mathbf{t})\psi_n\Gamma(\mathbf{t})^{-1} = \sum_{m=0}^{\infty} \psi_{n-m}h_m(\mathbf{t})$ .

We want to mark out that apart of traditional notation  $s_\lambda$  we shall use also notation  $s_{\{h\}}$  widely used in physical literature, say for instance [83]. Here  $h_i = \lambda_i - i + N$  where  $N$  is the length of a partition  $\lambda$ .

## B Charged and neutral free fermions.

For the charged fermions we shall use notations and conventions adopted in [6]. In particular for charged fermions we have

$$\langle 0 | \psi_{-n-1}^\dagger | = \langle 0 | \psi_n = 0, \quad \psi_{-n-1} | 0 \rangle = \psi_n^\dagger | 0 \rangle = 0, \quad n \geq 0 \quad (328)$$

$$\psi_m^\dagger \psi_n + \psi_n \psi_m^\dagger = \delta_{nm}, \quad n \in \mathbb{Z} \quad (329)$$

Such fermions are used to construct tau functions of KP, Toda lattice (TL) and the large IDKP hierarchies.

For "small" BKP hierarchy and "small" BKP hierarchy coupled to IDKP we need neutral fermions,  $\{\phi_n\}_{n \in \mathbb{Z}}$ , see [6], defined by the following property:

$$\phi_n \phi_m + \phi_m \phi_n = (-1)^n \delta_{n, -m}, \quad n \in \mathbb{Z} \quad (330)$$

$$\phi_n \psi_m + \psi_m \phi_n = 0, \quad \phi_n \psi_m^\dagger + \psi_m^\dagger \phi_n = 0 \quad (331)$$

It results from (330) that  $(\phi_0)^2 = \frac{1}{2}$ .

The action of neutral fermions on vacuum states are defined by

$$\phi_n | 0 \rangle = 0, \quad \langle 0 | \phi_{-n} = 0, \quad n < 0, \quad (332)$$

$$\phi_0 | 0 \rangle = \frac{1}{\sqrt{2}} | 0 \rangle, \quad \langle 0 | \phi_0 = \frac{1}{\sqrt{2}} \langle 0 | \quad (333)$$

Note that the action of  $\phi_0$  on the vacuum vectors is different from the action defined in [6]. This causes a modification in formulation of the Wick's relations, see below. Here we follow [13], where the choice of the corresponding Fock space is different from the suggested in [6]. See also the Appendix in the arxiv version of [69] for some details.

For  $n, m \in \mathbb{Z}$  we have

$$\langle 0 | \psi_n \phi_m | 0 \rangle = \langle 0 | \psi_n^\dagger \phi_m | 0 \rangle = 0$$

The right vacuum vector of a charge  $l$  is defined via

$$|l\rangle = \begin{cases} \psi_{l-1} \cdots \psi_0 | 0 \rangle & \text{if } l > 0 \\ \psi_l^\dagger \cdots \psi_{-1}^\dagger | 0 \rangle & \text{if } l < 0 \end{cases} \quad (334)$$

The dual vector is defined as

$$\langle l| = \begin{cases} \langle 0 | \psi_0^\dagger \cdots \psi_{l-1}^\dagger & \text{if } l > 0 \\ \langle 0 | \psi_{-1} \cdots \psi_l & \text{if } l < 0 \end{cases} \quad (335)$$

**Basis Fock vectors.** We shall use the following notation

$$|\lambda, l\rangle = (-)^{\beta_1 + \cdots + \beta_k} \psi_{\alpha_1 + l} \cdots \psi_{\alpha_k + l} \psi_{l - \beta_k - 1}^\dagger \cdots \psi_{l - \beta_1 - 1}^\dagger | l \rangle \quad (336)$$

where  $\lambda = (\alpha_1, \dots, \alpha_k | \beta_1, \dots, \beta_k)$ . Dual vectors may be defined via  $\langle \lambda', l' | \lambda, l \rangle = \delta_{\lambda, \lambda'} \delta_{l, l'}$ . The vector (336) may be also written as

$$|\lambda, l\rangle = \psi_{h_1 + l - N} \cdots \psi_{h_N + l - N} | l - N \rangle \quad (337)$$

where  $\lambda = (\lambda_1, \dots, \lambda_N)$  and

$$h_i := \lambda_i - i + N, \quad i = 1, \dots, N \quad (338)$$

which are called shifted parts of  $\lambda$ . Here  $N > \beta_1$ .

**Wick's relations.** Let each of  $w_i$  be a linear combination of Fermi operators:

$$w_i = \sum_{m \in \mathbb{Z}} w_{im}^- \psi_m + \sum_{m \in \mathbb{Z}} w_{im}^+ \psi_m^\dagger + \sum_{m \in \mathbb{Z}} w_{im} \phi_m, \quad i = 1, \dots, n$$

If  $n$  is even

$$\langle l | w_1 \cdots w_n | l \rangle = \text{Pf} [A] \quad (339)$$

where  $A$  is  $n$  by  $n$  antisymmetric matrix with entries

$$A_{ij} = \langle l | w_i w_j | l \rangle, \quad i < j$$

Now turn to the case where  $n$  is odd. In case zero mode operator  $\phi_0$  is absent in series for  $w_i$ , namely each  $w_{i0} = 0$ , we have the standard situation where  $\langle l | w_1 \cdots w_n | l \rangle$  vanishes<sup>11</sup>.

## C Wick's rule and Pfaffian representations

From the fermionic expression for tau functions we obtain its Pfaffian representations as a result of application of the Wick theorem.

First we suppose that  $g$  may be factorized into the product  $g = g^- g^+$  such that

$$g = g^- g^+ : \quad \langle l' | g^- = \langle l' |, \quad g^+ | l \rangle = | l \rangle \quad (340)$$

Let each  $w_i$ ,  $i = 1, \dots, 2m$  is a linear combination

$$w_i = \sum_{n \in \mathbb{Z}} (\alpha_{in} \psi_n + \beta_{in} \psi_n^\dagger) + \sum_{n \in \mathbb{Z}} \gamma_{in} \phi_n$$

---

<sup>11</sup>We exclude  $\phi_0$  because we use Kac-de Leur Fock space [13] where  $\langle 0 | \phi_0 | 0 \rangle = 1$  non-vanishes, see Appendix E.1

If  $n + m = 2k$  by Wick theorem we have

$$\langle l | w_1 \cdots w_n g w_{n+1} \cdots w_{2m} | l \rangle = \text{Pf}[W_{ij}]_{i,j=1,\dots,2k} \quad (341)$$

where  $W$  is the  $2k \times 2k$  antisymmetric block-structured matrix with entries given as follows

$$W_{ij} = \langle l | w_i w_j g | l \rangle, \quad W_{ip} = \langle l | w_i g w_p | l \rangle, \quad W_{pq} = \langle l | g w_p w_q | l \rangle \quad (342)$$

where  $1 \leq i < j < p < q \leq n$ .

## D On the central extension in Lie algebra of $\Psi DO$

The central extension in the algebra of  $\Psi DO$  may be chosen via the choice of nontrivial 2-cocycle in this algebra. It is known that there two independent cocycles found in [95]:

$$\omega_1(A, B) = \text{res}_x \text{res}_{\partial_x} (A[\log x, B]), \quad \omega_2(A, B) = \text{res}_x \text{res}_{\partial_x} (A[\log \partial_x, B])$$

In our case the choice of the cocycle should reproduce the Japanese cocycle in the algebra of infinite Jacobian matrices [6] and it should be chosen as follows

$$\omega(A, B) = \text{res}_x \text{res}_{D_x} (x^{-1} A[\log D_x, B]), \quad A, B \in \Psi DO$$

where the commutator of  $\log D_x$  with a  $\Psi DO$  is defined via relations

$$[\ln D_x, f(D_x)] = 0, \quad [\log D_x, f(x)] := \sum_{n=1}^{\infty} (-)^{n+1} \frac{(D_x^n \cdot f(x)) D_x^{-n}}{n}$$

Up to coboundary term  $\omega = \omega_1 + \omega_2$ .

## E Hirota equations

Fermionic form of Hirota equations was invented in the papers of Kyoto school, see for instance [6] and references therein.

Introduce

$$S_1 := \sum_{n \neq 0} (-)^n \phi_n \otimes \phi_{-n}, \quad S_2 := \sum_{n \neq 0} (-)^n \hat{\phi}_n \otimes \hat{\phi}_{-n} \quad (343)$$

$$S_0 := \phi_0 \otimes \phi_0 \quad (344)$$

$$S_3 := \sum_{n \in \mathbb{Z}} \psi_n \otimes \psi_n^\dagger, \quad S_4 := \sum_{n \in \mathbb{Z}} \psi_n^\dagger \otimes \psi_n, \quad (345)$$

Hirota equations in the fermionic form are:

For KP hierarchy [6]:

$$[g \otimes g, S_3] = 0 \quad (346)$$

For large (the same fermionic) DKP hierarchy [13]:

$$[g \otimes g, S_3 + S_4] = 0 \quad (347)$$

For large (fermionic) BKP hierarchy [13]:

$$[g \otimes g, S_3 + S_4 + S_0] = 0 \quad (348)$$

For small DKP hierarchy [4] (see also [13]):

$$[g \otimes g, S_1] = 0 \quad \text{or} \quad [g \otimes g, S_1] = 0 \quad (349)$$

For small BKP hierarchy [4] (see also [13]):

$$[g \otimes g, S_1 + S_0] = 0 \quad (350)$$

For two-component BKP hierarchy in form [13]:

$$[g \otimes g, S_1 + S_2 + S_0] = 0 \quad (351)$$

## E.1 A remarks on BKP hierarchies [13] and [6] and related vacuum expectation values

Let us note that different vacuum states were used in the constructions of BKP hierarchy in versions [13] and [6]. If we denote the left and right vacuum states used in [6] respectively by  $'\langle 0|$  and  $|0\rangle'$  then

$$\langle 0| = \frac{1}{\sqrt{2}} '\langle 0| + '\langle 0|\phi_0, \quad |0\rangle = \frac{1}{\sqrt{2}} |0\rangle' + \phi_0|0\rangle' \quad (352)$$

Introduce also

$$' \langle 1| = \sqrt{2} '\langle 0|\phi_0, \quad |1\rangle' = \sqrt{2} \phi_0|0\rangle', \quad (353)$$

then  $'\langle 0||0\rangle' = '\langle 1||1\rangle' = 1$  and instead of (332) we have

$$\phi_n|0\rangle' = \phi_n|1\rangle' = 0, \quad '\langle 0|\phi_{-n} = '\langle 1|\phi_{-n} = 0, \quad n < 0 \quad (354)$$

see [6] for details.

Correspondingly Fock spaces used [13] and [6] are different. From the representational point of view this definition is somewhat more convenient, since each Fock module remains irreducible for the algebra  $B_\infty$  which is the underlying algebra for KP equations of type B (BKP), see [13].

The vacuum states  $'\langle 0|$  and  $|0\rangle'$  are more familiar objects in physics. In particular any vacuum expectation value of an odd number of fermions vanishes, while, for instance,  $\langle 0|\phi_0|0\rangle = \frac{1}{\sqrt{2}}$ .

Let  $F$  be a product of even number of fermions. Then it is easy to see that

$$\langle 0|F|0\rangle = '\langle 0|F|0\rangle' \quad (355)$$

## E.2 A remark on formulae containing $Q_\lambda$ functions

From [10] it is known that

$$' \langle 0|e^{H(s)}\phi_{\lambda_1}\phi_{\lambda_2}\cdots\phi_{\lambda_N}|0\rangle' = \begin{cases} 2^{-\frac{N}{2}}Q_{(\lambda_1,\lambda_2,\dots,\lambda_N)}(\frac{s}{2}) & \text{for } N \text{ even,} \\ 0 & \text{for } N \text{ odd,} \end{cases} \quad (356)$$

$$\sqrt{2}' \langle 0|\phi_0e^{H(s)}\phi_{\lambda_1}\phi_{\lambda_2}\cdots\phi_{\lambda_N}|0\rangle' = \begin{cases} 2^{-\frac{N}{2}}Q_{(\lambda_1,\lambda_2,\dots,\lambda_N)}(\frac{s}{2}) & \text{for } N \text{ odd,} \\ 0 & \text{for } N \text{ even,} \end{cases} \quad (357)$$

where  $Q_\lambda(\frac{s}{2})$ ,  $\lambda = (\lambda_1, \lambda_2, \dots)$  are the projective Schur functions, see [18]. Thus

$$\begin{aligned} \langle 0|e^{H(s)}\phi_{\lambda_1}\phi_{\lambda_2}\cdots\phi_{\lambda_N}|0\rangle &= \left(\frac{1}{\sqrt{2}}'\langle 0| + '\langle 0|\phi_0\right)e^{H(s)}\phi_{\lambda_1}\phi_{\lambda_2}\cdots\phi_{\lambda_N}\left(\frac{1}{\sqrt{2}}|0\rangle' + \phi_0|0\rangle'\right) = \\ &= \frac{1}{2}'\langle 0|e^{H(s)}\phi_{\lambda_1}\phi_{\lambda_2}\cdots\phi_{\lambda_N}|0\rangle' + '\langle 0|\phi_0e^{H(s)}\phi_{\lambda_1}\phi_{\lambda_2}\cdots\phi_{\lambda_N}\phi_0|0\rangle' + \\ &+ \frac{1}{\sqrt{2}}'\langle 0|\phi_0e^{H(s)}\phi_{\lambda_1}\phi_{\lambda_2}\cdots\phi_{\lambda_N}|0\rangle' + \frac{1}{\sqrt{2}}'\langle 0|e^{H(s)}\phi_{\lambda_1}\phi_{\lambda_2}\cdots\phi_{\lambda_N}\phi_0|0\rangle' \\ &= \frac{1}{2}'\langle 0|e^{H(s)}\phi_{\lambda_1}\phi_{\lambda_2}\cdots\phi_{\lambda_N}|0\rangle' + \frac{1}{2}'\langle 1|e^{H(s)}\phi_{\lambda_1}\phi_{\lambda_2}\cdots\phi_{\lambda_N}|1\rangle' \\ &+ \frac{1}{\sqrt{2}}'\langle 0|\phi_0e^{H(s)}\phi_{\lambda_1}\phi_{\lambda_2}\cdots\phi_{\lambda_N}|0\rangle' + \frac{1}{\sqrt{2}}'\langle 1|\phi_0e^{H(s)}\phi_{\lambda_1}\phi_{\lambda_2}\cdots\phi_{\lambda_N}|1\rangle' \\ &= 2^{-\frac{N}{2}}Q_{(\lambda_1,\lambda_2,\dots,\lambda_N)}\left(\frac{s}{2}\right), \end{aligned}$$

since the role of  $'\langle 0|$  and  $'\langle 1|$  (resp.  $|0\rangle'$  and  $|1\rangle'$ ) is interchangeable.

### E.3 Projective Schur functions ( $Q$ -functions) and neutral fermions

There are two bases of *neutral free fermions*

$$\phi_i = \frac{1}{\sqrt{2}}(\psi_i + (-1)^i \psi_{-i}^*), \quad \hat{\phi}_i = \frac{i}{\sqrt{2}}(\psi_i - (-1)^i \psi_{-i}^*), \quad (358)$$

where  $i \in \mathbb{Z}$ , each of which generates this subalgebra.

Using the results for charged free fermions, the anticommutation relations are

$$[\phi_i, \phi_j]_+ = [\hat{\phi}_i, \hat{\phi}_j]_+ = (-1)^i \delta_{i,-j}, \quad [\phi_i, \hat{\phi}_j]_+ = 0, \quad (359)$$

and, in particular,  $\phi_0^2 = \hat{\phi}_0^2 = \frac{1}{2}$ . Similarly, the vacuum expectation values of quadratic elements are given by

$$' \langle 0 | \phi_i \phi_j | 0 \rangle ' = ' \langle 0 | \hat{\phi}_i \hat{\phi}_j | 0 \rangle ' = \begin{cases} (-1)^i \delta_{i,-j} & i < 0 \\ \frac{1}{2} \delta_{j,0} & i = 0 \\ 0 & i > 0 \end{cases}, \quad (360)$$

and Wick's Theorem is used for arbitrary degree products.

The neutral free fermion generator is defined by  $\phi(p) = \sum_{n \in \mathbb{Z}} p^n \phi_n$ . We have (for  $|p| \neq |p'|$ )

$$' \langle 0 | \phi(p) \phi(p') | 0 \rangle ' = \frac{1}{2} \frac{p - p'}{p + p'}, \quad (361)$$

and  $' \langle 0 | \phi(p') \phi(p) | 0 \rangle ' = -' \langle 0 | \phi(p) \phi(p') | 0 \rangle '$ . By Wick's Theorem we get

$$' \langle 0 | \phi(p_1) \phi(p_2) \cdots \phi(p_N) | 0 \rangle ' = \begin{cases} \text{Pf} [ ' \langle 0 | \phi(p_i) \phi(p_j) | 0 \rangle ' ] & N \text{ even} \\ 0 & \text{otherwise} \end{cases} = \begin{cases} 2^{-N/2} \prod_{i < j} \frac{p_i - p_j}{p_i + p_j} & N \text{ even} \\ 0 & \text{otherwise} \end{cases}. \quad (362)$$

The connection between the charged and neutral free fermions can be expressed in terms of the generators as

$$-q\psi(p)\psi^*(-q) + p\psi(q)\psi^*(-p) = \phi(p)\phi(q) + \hat{\phi}(p)\hat{\phi}(q). \quad (363)$$

In the sBKP reduction, even times are set equal to zero and we define  $\mathbf{t}' = (t'_1, 0, t'_3, 0, t'_5, \dots)$ , and the hamiltonian

$$H^B(\mathbf{t}') = \sum_{n \geq 1, \text{ odd}} H_n^B t'_n, \quad (364)$$

where

$$H_n^B = \frac{1}{2} \sum_{i \in \mathbb{Z}} (-1)^{i+1} \phi_i \phi_{-i-n}. \quad (365)$$

For the fermion generating function one has

$$\phi(p)(\mathbf{t}') = e^{H^B(\mathbf{t}')} \phi(p) e^{-H^B(\mathbf{t}')} = e^{\hat{H}^B(\mathbf{t}')} \phi(p) e^{-\hat{H}^B(\mathbf{t}')} = e^{\xi(p, \mathbf{t}')} \phi(p). \quad (366)$$

Note also that

$$H(\mathbf{t}') = H^B(\mathbf{t}') + \hat{H}^B(\mathbf{t}'), \quad [H^B(\mathbf{t}'), \hat{H}^B(\mathbf{t}')] = 0. \quad (367)$$

Similar to the KP case, sBKP  $\tau$ -functions are defined by

$$\tau_B(\mathbf{t}') = \langle h(\mathbf{t}') \rangle, \quad (368)$$

where  $h$  is the Clifford algebra of the neutral free fermions  $\phi_i$ . The  $n$ -soliton  $\tau$ -function is obtained by the choice  $g = \exp(\sum_{i=1}^n a_i \phi(p_i) \phi(q_i))$ .

The Schur  $q$  polynomials are defined by

$$\exp(2\xi(p, \mathbf{t}')) = \sum_{k \geq 0} q_k(\mathbf{t}') p^k. \quad (369)$$

Thus

$$\phi_i(\mathbf{t}') = \sum_{k \geq 0} q_k(\frac{1}{2}\mathbf{t}') \phi_{i-k}. \quad (370)$$

We have

$$\langle 0 | \phi_i(\mathbf{t}') \phi_j(\mathbf{t}') | 0 \rangle' = \frac{1}{2} q_i(\frac{1}{2}\mathbf{t}') q_j(\frac{1}{2}\mathbf{t}') + \sum_{k=1}^j (-1)^k q_{k+i}(\frac{1}{2}\mathbf{t}') q_{j-k}(\frac{1}{2}\mathbf{t}'). \quad (371)$$

Since

$$1 = \exp(2\xi(p, \mathbf{t}')) \exp(-2\xi(p, \mathbf{t}')) = \sum_{i,j} q_i(\mathbf{t}') q_{j-i}(-\mathbf{t}') = \sum_{i,j} (-1)^{i-j} q_i(\mathbf{t}') q_{j-i}(\mathbf{t}') p^j, \quad (372)$$

for all  $n > 0$  we have

$$\sum_{i=0}^n (-1)^i q_i(\mathbf{t}') q_{n-i}(\mathbf{t}') = 0. \quad (373)$$

This is trivial if  $n$  is odd and if  $n = 2m$  is even then it gives

$$q_m(\mathbf{t}')^2 + 2 \sum_{k=1}^m (-1)^k q_{m+k}(\mathbf{t}') q_{m-k}(\mathbf{t}') = 0. \quad (374)$$

We can also define

$$q_{a,b}(\mathbf{t}') = q_a(\mathbf{t}') q_b(\mathbf{t}') + 2 \sum_{k=1}^b (-1)^k q_{a+k}(\mathbf{t}') q_{b-k}(\mathbf{t}'). \quad (375)$$

It follows from the orthogonality condition (373) that

$$q_{a,b}(\mathbf{t}') = -q_{b,a}(\mathbf{t}'), \quad (376)$$

and in particular,  $q_{a,a}(\mathbf{t}') = 0$ . Comparing (371) and (375), it is clear that

$$q_{a,b}(\frac{1}{2}\mathbf{t}') = 2 \langle \phi_a(\mathbf{t}') \phi_b(\mathbf{t}') \rangle. \quad (377)$$

Now consider  $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_{2n})$  where  $\lambda_1 > \lambda_2 > \dots > \lambda_{2n-1} > \lambda_{2n} \geq 0$ . Note that this is a partition with an extra trivial part 0 included if necessary to ensure that the number of parts is even. The set of such strict, or distinct part, partitions is denoted DP. For  $\lambda \in \text{DP}$  we define

$$Q_\lambda(\frac{1}{2}\mathbf{t}') = \text{Pf}(q_{\lambda_i, \lambda_j}(\frac{1}{2}\mathbf{t}')). \quad (378)$$

This is the Schur  $Q$ -function. By Wick's theorem, we come to

**Lemma 2.**

$$Q_\lambda(\frac{1}{2}\mathbf{t}') = \text{Pf}(2 \langle \phi_{\lambda_i}(\mathbf{t}') \phi_{\lambda_j}(\mathbf{t}') \rangle) = 2^n \langle \phi_{\lambda_1}(\mathbf{t}') \phi_{\lambda_2}(\mathbf{t}') \dots \phi_{\lambda_{2n}}(\mathbf{t}') \rangle.$$

This wonderful result was obtained in [10] (see also [11] where it was independently found that  $Q_\lambda$  is an example of sBKP tau function). In particular, from this Lemma and (358), (327) it was obtained [10]

$$2^{-\frac{\ell(\lambda)}{2}} Q_\lambda(\frac{1}{2}\mathbf{t}') = \sqrt{s_{\tilde{\lambda}}(\mathbf{t}')}, \quad (379)$$

where  $s_{\tilde{\lambda}}(\mathbf{t})$  is the Schur function, and the partition  $\tilde{\lambda} \in P$  is the *double* (see 1.I, Ex9(a) of [18]) of the strict partition  $\lambda$ .  $\ell(\lambda)$  is the length of  $\lambda$  (the number of non-vanishing parts of  $\lambda$ ).



#### E.4 Schur functions and projective Schur functions evaluated at special values of its argument.

We introduce the following notations:

$$\mathbf{t}_\infty = (1, 0, 0, 0, \dots), \quad (380)$$

$$\mathbf{t}(a, 1) = \left(\frac{a}{1}, \frac{a}{2}, \frac{a}{3}, \dots\right), \quad (381)$$

$$\mathbf{t}(\infty, q) = (t_1(\infty, q), t_2(\infty, q), \dots), \quad t_m(\infty, q) = \frac{1}{m(1 - q^m)}, \quad m = 1, 2, \dots, \quad (382)$$

$$\mathbf{t}(a, q) = (t_1(a, q), t_2(a, q), \dots), \quad t_m(a, q) = \frac{1 - (q^a)^m}{m(1 - q^m)}, \quad m = 1, 2, \dots \quad (383)$$

For various purposes these choices of times were used in [29], [30], [27], [31], [97], [96], [32]. Note that  $\mathbf{t}(a, q)$  tends to  $\mathbf{t}(\infty, q)$  (resp.  $\mathbf{t}(a, 1)$ ) as  $a \rightarrow \infty$  (resp.  $q \rightarrow 1$ ). As for  $\mathbf{t}_\infty$ , if  $f$  satisfies  $f(ct_1, c^2t_2, c^3t_3, \dots) = c^d f(t_1, t_2, t_3, \dots)$  for some  $d \in \mathbb{Z}$ , we have  $\hbar^d f(\mathbf{t}(\infty, q)) \rightarrow f(\mathbf{t}_\infty)$  as  $\hbar := \ln q \rightarrow 0$ .

**Lemma 3.** For a partition  $\lambda = (\lambda_1, \lambda_2, \dots)$ , let  $h_i := n + \lambda_i - i$  ( $1 \leq i \leq n$ ), where  $n \geq \ell(\lambda)$ . Then

$$s_\lambda(\mathbf{t}_\infty) = \frac{1}{H_\lambda} = \frac{\Delta(h)}{\prod_{i=1}^n h_i!}, \quad (384)$$

$$s_\lambda(\mathbf{t}(a, 1)) = \frac{(a)_\lambda}{H_\lambda} = \frac{\Delta(h)}{\prod_{i=1}^n h_i!} \prod_{i=1}^n \frac{\Gamma(a - n + h_i + 1)}{\Gamma(a - i + 1)}, \quad (385)$$

$$s_\lambda(\mathbf{t}(\infty, q)) = \frac{q^{n(\lambda)}}{H_\lambda(q)} = \frac{\Delta(q^h)}{\prod_{i=1}^n (q; q)_{h_i}}, \quad (386)$$

$$s_\lambda(\mathbf{t}(a, q)) = \frac{q^{n(\lambda)}(q^a; q)_\lambda}{H_\lambda(q)} = c_n(a, q) \frac{\Delta(q^h)}{\prod_{i=1}^n (q; q)_{h_i}} \prod_{i=1}^n (q^a; q)_{h_i - n + 1}, \quad (387)$$

$$\Delta(h) := \prod_{i < j}^n (h_i - h_j), \quad \Delta(q^h) := \prod_{i < j}^n (q^{h_i} - q^{h_j}), \quad c_n(a, q) = \prod_{i=1}^n (1 - q^{a-i})^{n-i}$$

where for  $H_\lambda$ ,  $H_\lambda(q)$ ,  $n(\lambda)$ ,  $(a)_\lambda$  and  $(q^a; q)_\lambda$  see respectively (317), (318), (319), (320) and (321). Note that those quantities (384)–(387) are independent of the choice of  $n \geq \ell(\lambda)$ .

**Lemma 4.** . Let  $(\alpha|\beta) = (\alpha_1, \dots, \alpha_k | \beta_1, \dots, \beta_k)$  be the Frobenius notation for a partition. Then,

$$s_{(\alpha|\beta)}(\mathbf{t}_\infty) = \frac{\prod_{i < j}^k (\alpha_i - \alpha_j)(\beta_i - \beta_j)}{\prod_{i,j=1}^k (\alpha_i + \beta_j + 1)} \frac{1}{\prod_{i=1}^k \alpha_i! \prod_{i=1}^k \beta_i!}, \quad (388)$$

$$s_{(\alpha|\beta)}(\mathbf{t}(a, 1)) = \frac{\prod_{i < j}^k (\alpha_i - \alpha_j)(\beta_i - \beta_j)}{\prod_{i,j=1}^k (\alpha_i + \beta_j + 1)} \prod_{i=1}^k \frac{(a)_{\alpha_i+1}}{\alpha_i!} \prod_{i=1}^k \frac{(-)^{\beta_i} (-a)_{\beta_i}}{\beta_i!}, \quad (389)$$

$$s_{(\alpha|\beta)}(\mathbf{t}(\infty, q)) = \frac{\prod_{i < j}^k (q^{\alpha_i+1} - q^{\alpha_j+1})(q^{-\beta_j} - q^{-\beta_i})}{\prod_{i,j=1}^k (q^{-\beta_i} - q^{\alpha_j+1})} \frac{1}{\prod_{i=1}^k (q; q)_{\alpha_i} \prod_{i=1}^k (q; q)_{\beta_i}}, \quad (390)$$

$$s_{(\alpha|\beta)}(\mathbf{t}(a, q)) = \quad (391)$$

$$\frac{\prod_{i < j}^k (q^{\alpha_i+1} - q^{\alpha_j+1})(q^{-\beta_j} - q^{-\beta_i})}{\prod_{i,j=1}^k (q^{-\beta_i} - q^{\alpha_j+1})} \prod_{i=1}^k \frac{(q^a; q)_{\alpha_i+1}}{(q; q)_{\alpha_i}} \prod_{i=1}^k \frac{(-)^{\beta_i} q^{(a-1)\beta_i} (q^{1-a}; q)_{\beta_i}}{(q; q)_{\beta_i}} \quad (392)$$

From (379) and from (388) it may be derived

**Lemma 5.** Let  $\lambda = (\lambda_1, \dots, \lambda_k)$  be a strict partition. Then

$$Q_\lambda(\tfrac{1}{2}\mathbf{t}_\infty) = 2^{\frac{\ell(\lambda)}{2}} \prod_{i=1}^k \frac{1}{\lambda_i!} \prod_{i < j}^k \frac{\lambda_i - \lambda_j}{\lambda_i + \lambda_j}, \quad (393)$$

## E.5 "Neutral" two-component BKP hierarchy and 2-BKP hierarchy

**Two-component sBKP.** Consider the following tau function of the "small" two-component BKP hierarchy

$$\tau^{(B)}(\mathbf{t}', \mathbf{t}'', C) = \langle 0, 0 | \Gamma_B(\mathbf{t}') \hat{\Gamma}_B(\mathbf{t}'') \exp \sum_{n,m \geq 0} C_{nm} \phi_n \hat{\phi}_m | 0, 0 \rangle \quad (394)$$

$$= \sum_{\lambda \in \mathbb{P}} 2^{-\frac{1}{2}\ell(\alpha) - \frac{1}{2}\ell(\beta)} \det C_\lambda Q_\alpha \left( \frac{\mathbf{t}'}{2} \right) Q_\beta \left( \frac{\mathbf{t}''}{2} \right) \quad (395)$$

where  $\lambda = (\alpha|\beta)$  and  $C_\lambda := \det (C)_{\alpha_i, \beta_j}$  and  $Q_\alpha(\frac{\mathbf{t}}{2})$  is the projective Schur function related to a (strict) partition  $\alpha$ , see [18]. Neutral fermions  $\phi_n, \hat{\phi}_n$  were considered in (358). If we take  $C_{nm} = C_{nm}(\mathbf{t}) = e^{U_m - U_n} s_{(n|m)}(\mathbf{t})$  we obtain

$$\tau^{(B)}(\mathbf{t}', \mathbf{t}'', C(\mathbf{t})) = \sum_{\lambda \in \mathbb{P}} 2^{-\ell(\lambda)} e^{U_{\{\beta\}} - U_{\{\alpha\}}} Q_\alpha \left( \frac{\mathbf{t}'}{2} \right) Q_\beta \left( \frac{\mathbf{t}''}{2} \right) s_\lambda(\mathbf{t}) \quad (396)$$

Thus this tau function is a tau function of three hierarchies at the same time which are the lDKP and two "small" BKP ones.

Let us present the following 'symmetric' fermionic representation of this tau function which looks little in common with usual notation for tau functions

$$\langle 0, 0, 0 | \Gamma(\mathbf{t}) \Gamma_B(\mathbf{t}') \Gamma_B(\mathbf{t}'') e^{\sum_{n,m \geq 0} e^{U_m - U_n} \psi_n \psi_{-m-1}^\dagger \varphi_n \hat{\varphi}_m} | 0, 0, 0 \rangle \quad (397)$$

**2-sBKP tau function.** This is

$$\tau(\mathbf{t}', \bar{\mathbf{t}}', A) = \langle 0 | \Gamma_B(\mathbf{t}') e^{\sum_{n,m} A_{nm} \phi_n \phi_m} \bar{\Gamma}_B(\bar{\mathbf{t}}') | 0 \rangle \quad (398)$$

where  $A$  is an anti-symmetric matrix. Thanks to

$$\langle 0 | \Gamma_B(\mathbf{t}') = \sum_{\alpha \in \text{DP}} \langle \alpha | 2^{-\frac{1}{2}\ell(\alpha)} Q_\alpha, \quad \bar{\Gamma}_B(\bar{\mathbf{t}}') | 0 \rangle = \sum_{\beta \in \text{DP}} | \beta \rangle 2^{-\frac{1}{2}\ell(\beta)} Q_\beta, \quad \langle \alpha | \beta \rangle = \delta_{\alpha, \beta} \quad (399)$$

by Wick theorem it may be written in form of double series over strict partitions as

$$= \sum_{\alpha, \beta \in \text{DP}} 2^{-\frac{1}{2}\ell(\alpha) - \frac{1}{2}\ell(\beta)} A_{\alpha, \beta} Q_\alpha(\mathbf{t}') Q_\beta(\bar{\mathbf{t}}') \quad (400)$$

where

$$A_{\alpha, \beta} = \det [A_{\alpha_n, \beta_m}], \quad A_{nm} = \langle 0 | \phi_{-n} e^{\sum_{i,j} A_{ij} \phi_i \phi_j} \phi_m | 0 \rangle \quad (401)$$

## F Appendix. Simplest lDKP solitons

We want to present certain types of solitonic solutions typical for lDKP hierarchy.

I. KP hierarchy is a reduction of lDKP one, therefore KP solitons are also lDKP ones. First let me remind a typical KP two-soliton tau function (which is also lDKP two-soliton tau function which will be denoted by  $\tau_{2sol}^I(\mathbf{t})$ )

$$\tau_{2sol}^I(\mathbf{t}) = \langle 0 | \Gamma(\mathbf{t}) e^{a_1 \psi(p_1) \psi^\dagger(q_1) + a_2 \psi(p_2) \psi^\dagger(q_2)} | 0 \rangle = \quad (402)$$

$$1 + e^{\delta_1} e^{\sum_{m=1}^\infty t_m (p_1^m - q_1^m)} + e^{\delta_2} e^{\sum_{m=1}^\infty t_m (p_2^m - q_2^m)} + e^{\delta_{12}} \prod_{i=1,2} e^{\delta_i \sum_{m=1}^\infty t_m (p_i^m - q_i^m)} \quad (403)$$

where

$$\delta_i = \log \frac{a_i}{p_i - q_i}, \quad \delta_{ij} = \frac{(p_i - p_j)(q_i - q_j)}{(p_i - q_j)(p_j - q_i)} \quad (404)$$

This two-soliton solution exhibit resonance behavior [43] when  $p_1 \rightarrow p_2$ , the same occurs when  $q_1 \rightarrow q_2$  (such solitons has the Manakov's Y-form).

For lDKP one can present a larger amount of various one- and two-soliton tau functions. Examples are written down below.

II. One of them we may obtain by replacing the right vacuum vector  $|0\rangle$  by  $|\Omega\rangle$  in the expectation in the right hand side of (402):

$$\tau_{2sol}^{II}(\mathbf{t}) = \langle 0 | \Gamma(\mathbf{t}) e^{a_1 \psi(p_1) \psi^\dagger(q_1) + a_2 \psi(p_2) \psi^\dagger(q_2)} | \Omega \rangle = \quad (405)$$

$$\tau_0(\mathbf{t}) \left( 1 + e^{\delta_1 + \Delta_1} e^{\sum_{m=1}^{\infty} t_m (p_1^m - q_1^m)} + e^{\delta_2 + \Delta_2} e^{\sum_{m=1}^{\infty} t_m (p_2^m - q_2^m)} + e^{\delta_{12} + \Delta_{12}} \prod_{i=1,2} e^{\delta_i + \Delta_i} e^{\sum_{m=1}^{\infty} t_m (p_i^m - q_i^m)} \right) \quad (406)$$

where  $\delta_{1,2}$  and  $\delta_{12}$  are given by (404) and where

$$\Delta_i = \log \frac{1}{(1-p_i)(1+q_i)}, \quad \Delta_{ij} = \log \frac{(1-p_i q_j)(1-p_j q_i)}{(1-p_i p_j)(1-q_i q_j)} \quad (407)$$

This solution describes interaction of two solitons,  $i = 1, 2$ , each is

$$\tau_{1sol}^{II}(\mathbf{t}) = \tau_0(\mathbf{t}) \left( 1 + e^{\delta_i + \Delta_i} e^{\sum_{m=1}^{\infty} t_m (p_i^m - q_i^m)} \right), \quad i = 1, 2 \quad (408)$$

(notice the  $\tau_0(\mathbf{t})$  factor in front of the right-hand side).

Now two-soliton solution exhibit resonance behavior also in case when  $p_1 \rightarrow q_2^{-1}$ , the same occurs when  $p_2 \rightarrow q_1^{-1}$ .

One can say that we add 'KP solitons' to lBKP background solution (222).

III. Adding of 'KP solitons' to a lBKP background solution  $\tau_0(\mathbf{t})$  of form (204) yields

$$\tau_{1sol}^{III}(\mathbf{t}) = \langle N + l | \Gamma(\mathbf{t}) e^{a \psi(p) \psi^\dagger(q)} g^{-l} | l \rangle = \tau_0(\mathbf{t}) \left( 1 + a p^l q^{-l} \right) \quad (409)$$

IV. lDKP 1-soliton solution may be also as follows

$$\tau_{1sol}^{IV}(\mathbf{t}) = \langle 0 | \Gamma(\mathbf{t}) e^{a_1 \psi(p_1) \psi(p_2) + a_2 \psi^\dagger(q_1) \psi^\dagger(q_2)} | 0 \rangle = 1 + e^{\delta_{12}} \prod_{i=1,2} e^{\delta_i} e^{\sum_{m=1}^{\infty} t_m (p_i^m - q_i^m)} \quad (410)$$

This soliton is characterizes by four spectral parameters  $p_1, p_2, q_1, q_2$  instead of a pair unlike, say, typical KP soliton.

The corresponding '2-soliton' tau function is

$$\tau_{2sol}^{III}(\mathbf{t}) = \langle 0 | \Gamma(\mathbf{t}) e^{a_{12} \psi(p_1) \psi(p_2) + a_{12}^* \psi^\dagger(q_1) \psi^\dagger(q_2) + a_{34} \psi(p_3) \psi(p_4) + a_{34}^* \psi^\dagger(q_3) \psi^\dagger(q_4)} | 0 \rangle = \quad (411)$$

$$1 + e^{\varepsilon_{12,12}} \prod_{i=1,2} e^{\sum_{m=1}^{\infty} p_i^m t_m} \prod_{i=1,2} e^{-\sum_{m=1}^{\infty} q_i^m t_m} + e^{\varepsilon_{34,34}} \prod_{i=3,4} e^{\sum_{m=1}^{\infty} p_i^m t_m} \prod_{i=3,4} e^{-\sum_{m=1}^{\infty} q_i^m t_m} \quad (412)$$

$$+ e^{\varepsilon_{12,34}} \prod_{i=1,2} e^{\sum_{m=1}^{\infty} p_i^m t_m} \prod_{i=3,4} e^{-\sum_{m=1}^{\infty} q_i^m t_m} + e^{\varepsilon_{34,12}} \prod_{i=3,4} e^{\sum_{m=1}^{\infty} p_i^m t_m} \prod_{i=1,2} e^{-\sum_{m=1}^{\infty} q_i^m t_m} + \quad (413)$$

$$+ e^{\varepsilon_{1234}} \prod_{i=1,2,3,4} e^{\sum_{m=1}^{\infty} t_m (p_i^m - q_i^m)} \quad (414)$$

where

$$\varepsilon_{ij,kl} = -\log(p_i - q_k)(p_i - q_l)(p_j - q_k)(p_j - q_l) + a_{ij} + \log(p_i - p_j) + a_{kl}^* + \log(q_k - q_l) \quad (415)$$

and where

$$\varepsilon_{1234} = \frac{\prod_{i < j}^4 (p_i - p_j)(q_i - q_j)}{\prod_{i,j=1}^4 (p_i - q_j)} + \log(a_{12} a_{12}^* a_{34} a_{34}^*) \quad (416)$$

$$\tau^{IV}(\mathbf{t}) = \langle 0 | \Gamma(\mathbf{t}) e^{\sum_{n,m} C_{nm} \psi(p_n) \psi^\dagger(q_m)} | \Omega \rangle = \quad (417)$$

$$\tau_0(\mathbf{t}) \sum_{k=0}^{\infty} \sum_{m_1, \dots, m_k} \sum_{n_1, \dots, n_k} \prod_{i < j} \frac{\prod_{i < j \leq k} (p_{m_i} - p_{m_j})(q_{n_i} - q_{n_j})}{\prod_{i,j=1}^k (p_{m_i} - q_{n_j})} \quad (418)$$

V. In case  $N > 0$ :

One-soliton

$$\tau_{1sol}^V(\mathbf{t}) = \langle N | \Gamma(\mathbf{t}) e^{a\psi^\dagger(q_1)\psi^\dagger(q_2)} g^{--} | 0 \rangle \quad (419)$$

$$= \sum_{\ell(\lambda) \leq N} s_\lambda(\mathbf{t}) + a(q_1 q_2)^N \sum_{\ell(\lambda) \leq N+2} s_\lambda(\mathbf{t} + [q_1^{-1}] + [q_2^{-1}]) \quad (420)$$

Now, let us be interested in the case where

$$t_m = t_m(\mathbf{x}) = \frac{1}{m} \sum_{i=1}^N x_i^m \quad (421)$$

Then

$$\tau_{1sol}^V(\mathbf{t}(\mathbf{x})) = \tau_0(\mathbf{t}(\mathbf{x})) \left( 1 + \frac{1}{(1-q_1)(1-q_2)(1-q_1 q_2)} \prod_{i=1}^N \frac{1}{(1-q_1 x_i)(1-q_2 x_i)} \right) \quad (422)$$